Math 12 H Lesson Plan – Partial Fractions – Section 9.8 – Day 1

We can combine fractions with addition or subtraction by finding the common denominator. Partial fractions undoes that. It takes the combined fraction and breaks it down into its individual pieces.

$$\frac{1}{x-1} + \frac{1}{2x+1} = \frac{3x}{2x^2-x-1}$$

$$\frac{3x}{2x^2-x-1} = \frac{1}{x-1} + \frac{1}{2x+1}$$

To solve for partial fractions, we have to eventually solve a system of equations at some point.

Any Polynomial with real coefficients can be broken down into a series of linear and quadratic factors. $\rightarrow$ (ax^2+bx+c) and $(ax+b)$.

When you break up partial fractions the format is:

$$\frac{A}{ax+b} + \frac{B}{a_2x+b_2} + \frac{C}{a_3x+b_3} \quad \text{and}$$

$$\frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{a_2x^2+b_2x+c_2} + \frac{Ex+F}{a_3x^2+b_3x+c_3} \quad \text{etc.}$$

There are 4 cases that can happen

Case 1) Denominator is a product of distinct linear factors. (no x^2).

Ex: $\frac{x+4}{x^2-4x} = \boxed{\frac{x+4}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4}}$

To solve; Get rid of fraction by multiplying by L.C.D.

$$\frac{x+4}{x(x-4)} \cdot (x)(x-4) = \frac{A}{x} \cdot x(x-4) + \frac{B}{x-4}(x)(x-4)$$

Equation Becomes:

$$x + 4 = A(x-4) + B(x)$$

Distribute then group coefficients of various x- terms.

$$x+4 = Ax - 4A + Bx$$

$$x + 4 = Ax + Bx - 4A$$

$$1x + 4 = (A+B)x - 4A$$

Now write as a system of equations

$$\left.\begin{aligned} A+B &= 1 \\ -4A &= 4 \end{aligned}\right\}$$ Solve $A=-1$ and $B=2$.

CASE 2) Denominator has unique Quadratic Factors.

Ex: $\frac{2x^2-x+4}{x^3+4x} = \frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$

For Cases 3 and 4 Factors are repeated - either linear or quadratic.

Ex: $$\frac{5x+7}{x^3+2x^2-x-2} = \frac{5x+7}{x^2(x+2)-1(x+2)} = \frac{5x+7}{(x^2-1)(x+2)}$$

$$= \frac{5x+7}{(x+1)(x-1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(x+2)}$$ Mult by L.C.D

$$= 5x+7 = A(x-1)(x+2) + B(x+1)(x+2) + C(x+1)(x-1)$$

$$5x+7 = A(x^2+x-2) + B(x^2+3x+2) + C(x^2-1)$$

$$5x+7 = Ax^2+Ax-2A+Bx^2+3Bx+2B+Cx^2-C$$

$$5x+7 = (A+B+C)x^2 + (A+3B)x + (-2A+2B-C)$$

System = $\begin{aligned} A+B+C &= 0 \\ A+3B &= 5 \\ -2A+2B-C &= 7 \end{aligned}$ ← Solve $\begin{aligned} A+B+C &= 0 \\ 2B-C &= 5 \\ 4B+C &= 7 \end{aligned}$

$\begin{aligned} A+B+C &= 0 \\ 2B-C &= 5 \\ 6B &= 12 \end{aligned}$ $\Rightarrow$ $B=2$, $C=-1$, $A=-1$

Plug back in to show Fraction Decomposition.

$$\frac{5x+7}{x^3+2x^2-x-2} = \boxed{\frac{-1}{x+1} + \frac{2}{x-1} - \frac{1}{x+2}}$$

Ex: $\dfrac{2x^2-x+4}{x^3+4x} = \dfrac{2x^2-x+4}{x(x^2+4)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+4}$ Mult by L.C.D.

$= 2x^2-x+4 = A(x^2+4) + (Bx+C)(x)$

$= 2x^2-x+4 = Ax^2+4A + Bx^2+Cx$

$= 2x^2-x+4 = (A+B)(x^2) + C(x) + 4A$

System $= \left.\begin{array}{rl} A+B & = 2 \\ C & = -1 \\ 4A & = 4 \end{array}\right] \rightarrow C=-1,\ A=1,\ B=1$

$$\boxed{\dfrac{2x^2-x+4}{x(x^2+4)} = \dfrac{1}{x} + \dfrac{1x-1}{x^2+4}}$$

If factors are repeated:

Ex: $\dfrac{x^5-3x^2+12x-1}{x^3(x^2+x+1)(x^2+2)^3}$

$$\frac{x^5-3x^2+12x-1}{x^3(x^2+x+1)(x^2+2)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{(x^2+2)} + \frac{Fx+G}{(x^2+2)^2} + \frac{Hx+I}{(x^2+2)^3} + \frac{Jx+K}{(x^2+x+1)}$$

That would be a very large system to solve,