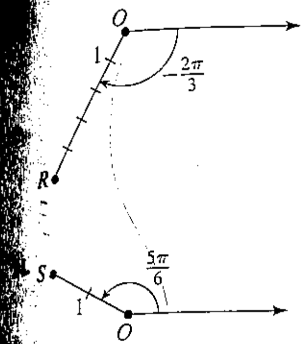
24 ■ A point is graphed in polar form. Find its rectangular dinates.



25–32 ■ Find the rectangular coordinates for the point whose polar coordinates are given.

25. $(4, \pi/6)$
26. $(6, 2\pi/3)$
27. $(\sqrt{2}, -\pi/4)$
28. $(-1, 5\pi/2)$
29. $(5, 5\pi)$
30. $(0, 13\pi)$
31. $(6\sqrt{2}, 11\pi/6)$
32. $(\sqrt{3}, -5\pi/3)$

33–40 ■ Convert the rectangular coordinates to polar coordinates with $r > 0$ and $0 \le \theta < 2\pi$.

33. $(-1, 1)$
34. $(3\sqrt{3}, -3)$
35. $(\sqrt{8}, \sqrt{8})$
36. $(-\sqrt{6}, -\sqrt{2})$
37. $(3, 4)$
38. $(1, -2)$
39. $(-6, 0)$
40. $(0, -\sqrt{3})$

41–46 ■ Convert the equation to polar form.

41. $x = y$
42. $x^2 + y^2 = 9$
43. $y = x^2$
44. $y = 5$
45. $x = 4$
46. $x^2 - y^2 = 1$

47–60 ■ Convert the polar equation to rectangular coordinates.

47. $r = 7$
48. $\theta = \pi$
49. $r \cos\theta = 6$
50. $r = 6\cos\theta$
51. $r^2 = \tan\theta$
52. $r^2 = \sin 2\theta$
53. $r = \dfrac{1}{\sin\theta - \cos\theta}$
54. $r = \dfrac{1}{1 + \sin\theta}$
55. $r = 1 + \cos\theta$
56. $r = \dfrac{4}{1 + 2\sin\theta}$
57. $r = 2\sec\theta$
58. $r = 2 - \cos\theta$
59. $\sec\theta = 2$
60. $\cos 2\theta = 1$

Discovery • Discussion

61. **The Distance Formula in Polar Coordinates**

(a) Use the Law of Cosines to prove that the distance between the polar points (r_1, θ_1) and (r_2, θ_2) is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$$

(b) Find the distance between the points whose polar coordinates are $(3, 3\pi/4)$ and $(1, 7\pi/6)$, using the formula from part (a).

(c) Now convert the points in part (b) to rectangular coordinates. Find the distance between them using the usual Distance Formula. Do you get the same answer?

8.2 Graphs of Polar Equations

The **graph of a polar equation** $r = f(\theta)$ consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation. Many curves that arise in mathematics and its applications are more easily and naturally represented by polar equations rather than rectangular equations.

A rectangular grid is helpful for plotting points in rectangular coordinates (see Figure 1(a) on the next page). To plot points in polar coordinates, it is conven-

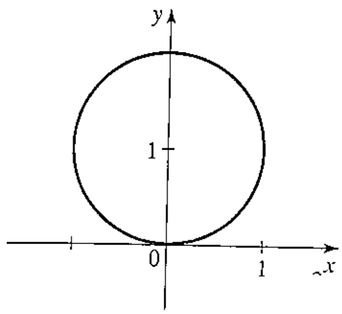


Figure 9

(b) We multiply both sides of the equation by r, because then we ca formulas $r^2 = x^2 + y^2$ and $r \sin\theta = y$.

$$r^2 = 2r\sin\theta \qquad \text{Multiply by } r$$

$$x^2 + y^2 = 2y \qquad r^2 = x^2 + y^2 \text{ and } r\sin\theta =$$

$$x^2 + y^2 - 2y = 0 \qquad \text{Subtract } 2y$$

$$x^2 + (y-1)^2 = 1 \qquad \text{Complete the square in } y$$

This is the equation of a circle of radius 1 centered at the point (0 graphed in Figure 9.

(c) We first multiply both sides of the equation by r:

$$r^2 = 2r + 2r\cos\theta$$

Using $r^2 = x^2 + y^2$ and $x = r\cos\theta$, we can convert two of the th the equation into rectangular coordinates, but eliminating the rem requires more work:

$$x^2 + y^2 = 2r + 2x \qquad r^2 = x^2 + y^2 \text{ and } r\cos\theta$$

$$x^2 + y^2 - 2x = 2r \qquad \text{Subtract } 2x$$

$$(x^2 + y^2 - 2x)^2 = 4r^2 \qquad \text{Square both sides}$$

$$(x^2 + y^2 - 2x)^2 = 4(x^2 + y^2) \qquad r^2 = x^2 + y^2$$

In this case, the rectangular equation looks more complicated than equation. Although we cannot easily determine the graph of the eq its rectangular form, we will see in the next section how to graph i polar equation.

8.1 Exercises

1–6 ■ Plot the point that has the given polar coordinates.

1. $(4, \pi/4)$ **2.** $(1, 0)$ **3.** $(6, -7\pi/6)$

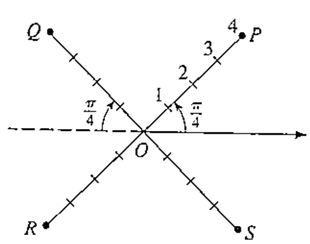
4. $(3, -2\pi/3)$ **5.** $(-2, 4\pi/3)$ **6.** $(-5, -17\pi/6)$

7–12 ■ Plot the point that has the given polar coordinates. Then give two other polar coordinate representations of the point, one with $r < 0$ and the other with $r > 0$.

7. $(3, \pi/2)$ **8.** $(2, 3\pi/4)$ **9.** $(-1, 7\pi/6)$

10. $(-2, -\pi/3)$ **11.** $(-5, 0)$ **12.** $(3, 1)$

13–20 ■ Determine which point in the figure, P, Q, R, or S, has the given polar coordinates.



13. $(4, 3\pi/4)$ **14.** $(4, -3\pi/4)$

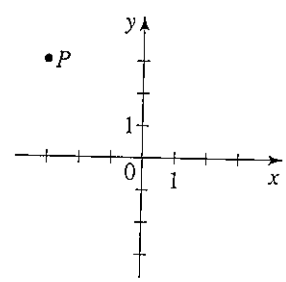
15. $(-4, -\pi/4)$ **16.** $(-4, 13\pi/4)$

17. $(4, -23\pi/4)$ **18.** $(-4, 23\pi/4)$

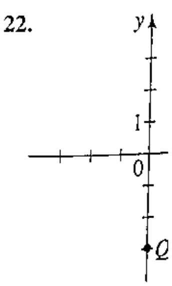
19. $(-4, 101\pi/4)$ **20.** $(4, 103\pi/4)$

21–22 ■ A point is graphed in rectangular form. F coordinates for the point, with $r > 0$ and $0 < \theta < 2$

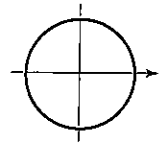
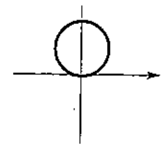
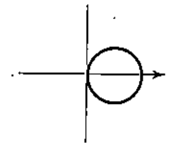
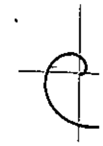
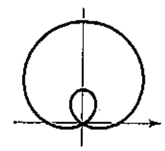
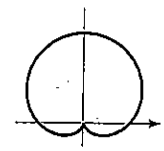
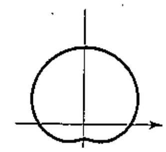
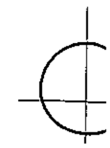
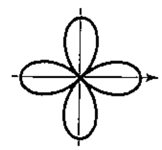
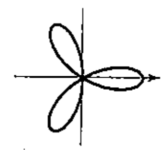
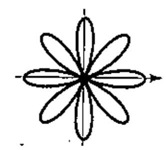
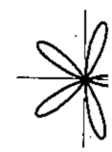
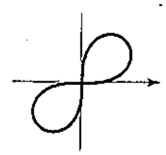
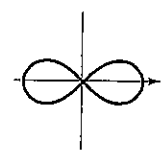
21.



22.



Some Common Polar Curves

Circles and Spiral	 $r = a$ circle	 $r = a \sin \theta$ circle	 $r = a \cos \theta$ circle	 $r = a$ spira
Limaçons $r = a \pm b \sin \theta$ $r = a \pm b \cos \theta$ $(a > 0, b > 0)$ Orientation depends on the trigonometric function (sine or cosine) and the sign of b.	 $a < b$ limaçon with inner loop	 $a = b$ cardioid	 $a > b$ dimpled limaçon	 $a \geq 2$ convex li
Roses $r = a \sin n\theta$ $r = a \cos n\theta$ n-leaved if n is odd $2n$-leaved if n is even	 $r = a \cos 2\theta$ 4-leaved rose	 $r = a \cos 3\theta$ 3-leaved rose	 $r = a \cos 4\theta$ 8-leaved rose	 $r = a$ co 5-leaved
Lemniscates Figure-eight-shaped curves	 $r^2 = a^2 \sin 2\theta$ lemniscate	 $r^2 = a^2 \cos 2\theta$ lemniscate		

8.2 Exercises

1–6 ■ Match the polar equation with the graphs labeled I–VI. Use the table above to help you.

1. $r = 3 \cos \theta$

2. $r = 3$

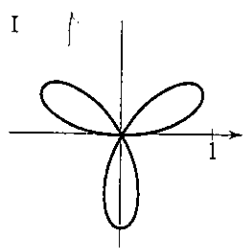
3. $r = 2 + 2 \sin \theta$

4. $r = 1 + 2 \cos \theta$

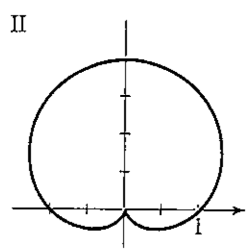
5. $r = \sin 3\theta$

6. $r = \sin 4\theta$

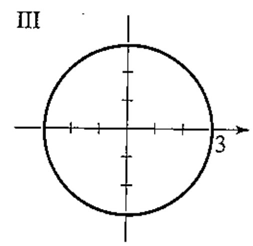
I



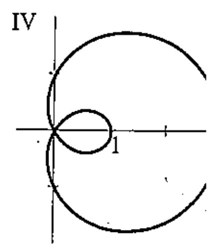
II



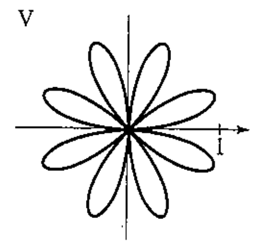
III



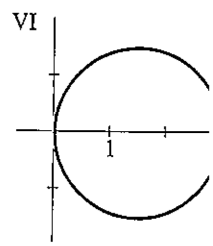
IV



V



VI



14 ■ Test the polar equation for symmetry with respect to the olar axis, the pole, and the line $\theta = \pi/2$.

7. $r = 2 - \sin\theta$
8. $r = 4 + 8\cos\theta$
9. $r = 3\sec\theta$
10. $r = 5\cos\theta\ \csc\theta$
1. $r = \dfrac{4}{3 - 2\sin\theta}$
12. $r = \dfrac{5}{1 + 3\cos\theta}$
3. $r^2 = 4\cos 2\theta$
14. $r^2 = 9\sin\theta$

5–36 ■ Sketch the graph of the polar equation.

5. $r = 2$
16. $r = -1$
7. $\theta = -\pi/2$
18. $\theta = 5\pi/6$
9. $r = 6\sin\theta$
20. $r = \cos\theta$
1. $r = -2\cos\theta$
22. $r = 2\sin\theta + 2\cos\theta$
3. $r = 2 - 2\cos\theta$
24. $r = 1 + \sin\theta$
5. $r = -3(1 + \sin\theta)$
26. $r = \cos\theta - 1$
7. $r = \theta,\quad \theta \ge 0$ (spiral)
8. $r\theta = 1,\quad \theta > 0$ (reciprocal spiral)
9. $r = \sin 2\theta$ (four-leaved rose)
0. $r = 2\cos 3\theta$ (three-leaved rose)
1. $r^2 = \cos 2\theta$ (lemniscate)
2. $r^2 = 4\sin 2\theta$ (lemniscate)
3. $r = 2 + \sin\theta$ (limaçon)
4. $r = 1 - 2\cos\theta$ (limaçon)
5. $r = 2 + \sec\theta$ (conchoid)
6. $r = \sin\theta\tan\theta$ (cissoid)

7–40 ■ Use a graphing device to graph the polar equation. hoose the domain of θ to make sure you produce the entire raph.

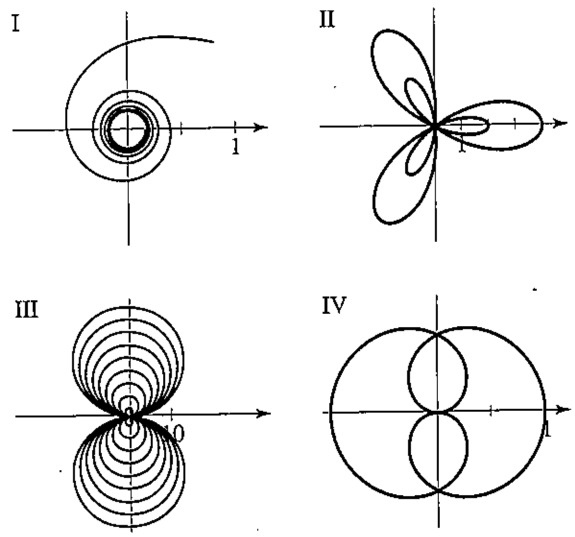
7. $r = \cos(\theta/2)$
38. $r = \sin(8\theta/5)$
9. $r = 1 + 2\sin(\theta/2)$ (nephroid)
0. $r = \sqrt{1 - 0.8\sin^2\theta}$ (hippopede)

1. Graph the family of polar equations $r = 1 + \sin n\theta$ for $n = 1, 2, 3, 4$, and 5. How is the number of loops related to n?

2. Graph the family of polar equations $r = 1 + c\sin 2\theta$ for $c = 0.3, 0.6, 1, 1.5$, and 2. How does the graph change as c increases?

3–46 ■ Match the polar equation with the graphs labeled I–IV. ive reasons for your answers.

3. $r = \sin(\theta/2)$
44. $r = 1/\sqrt{\theta}$
45. $r = \theta\sin\theta$
46. $r = 1 + 3\cos(3\theta)$



47–50 ■ Sketch a graph of the rectangular equation. [*Hint:* First convert the equation to polar coordinates.]

47. $(x^2 + y^2)^3 = 4x^2y^2$

48. $(x^2 + y^2)^3 = (x^2 - y^2)^2$

49. $(x^2 + y^2)^2 = x^2 - y^2$

50. $x^2 + y^2 = (x^2 + y^2 - x)^2$

51. Show that the graph of $r = a\cos\theta + b\sin\theta$ is a circle, and find its center and radius.

52. (a) Graph the polar equation $r = \tan\theta\sec\theta$ in the viewing rectangle $[-3, 3]$ by $[-1, 9]$.
(b) Note that your graph in part (a) looks like a parabola (see Section 2.5). Confirm this by converting the equation to rectangular coordinates.

Applications

53. **Orbit of a Satellite** Scientists and engineers often use polar equations to model the motion of satellites in earth orbit. Let's consider a satellite whose orbit is modeled by the equation $r = 22500/(4 - \cos\theta)$, where r is the distance in miles between the satellite and the center of the earth and θ is the angle shown in the figure on the next page.
(a) On the same viewing screen, graph the circle $r = 3960$ (to represent the earth, which we will assume to be a sphere of radius 3960 mi) and the polar equation of the satellite's orbit. Describe the motion of the satellite as θ increases from 0 to 2π.

10.7 Exercises

1–22 ■ A pair of parametric equations is given.

(a) Sketch the curve represented by the parametric equations.

(b) Find a rectangular-coordinate equation for the curve by eliminating the parameter.

1. $x = 2t, \quad y = t + 6$

2. $x = 6t - 4, \quad y = 3t, \quad t \geq 0$

3. $x = t^2, \quad y = t - 2, \quad 2 \leq t \leq 4$

4. $x = 2t + 1, \quad y = \left(t + \frac{1}{2}\right)^2$

5. $x = \sqrt{t}, \quad y = 1 - t$

6. $x = t^2, \quad y = t^4 + 1$

7. $x = \dfrac{1}{t}, \quad y = t + 1$

8. $x = t + 1, \quad y = \dfrac{t}{t + 1}$

9. $x = 4t^2, \quad y = 8t^3$

10. $x = |t|, \quad y = |1 - |t||$

11. $x = 2 \sin t, \quad y = 2 \cos t, \quad 0 \leq t \leq \pi$

12. $x = 2 \cos t, \quad y = 3 \sin t, \quad 0 \leq t \leq 2\pi$

13. $x = \sin^2 t, \quad y = \sin^4 t$

14. $x = \sin^2 t, \quad y = \cos t$

15. $x = \cos t, \quad y = \cos 2t$

16. $x = \cos 2t, \quad y = \sin 2t$

17 $x = \sec t, \quad y = \tan t, \quad 0 \leq t < \pi/2$

18 $x = \cot t, \quad y = \csc t, \quad 0 < t < \pi$

19 $x = \tan t, \quad y = \cot t, \quad 0 < t < \pi/2$

20. $x = \sec t, \quad y = \tan^2 t, \quad 0 \leq t < \pi/2$

21. $x = \cos^2 t, \quad y = \sin^2 t$

22. $x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \leq t \leq 2\pi$

23–26 ■ Find parametric equations for the line with the given properties.

23. Slope $\frac{1}{2}$, passing through $(4, -1)$

24. Slope -2, passing through $(-10, -20)$

25. Passing through $(6, 7)$ and $(7, 8)$

26. Passing through $(12, 7)$ and the origin

27. Find parametric equations for the circle $x^2 + y^2 = a^2$.

28. Find parametric equations for the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

29. Show by eliminating the parameter θ that the following parametric equations represent a hyperbola:

$$x = a \tan \theta \qquad y = b \sec \theta$$

30. Show that the following parametric equations represent a part of the hyperbola of Exercise 29:

$$x = a\sqrt{t} \qquad y = b\sqrt{t + 1}$$

31–34 ■ Sketch the curve given by the parametric equations.

31. $x = t \cos t, \quad y = t \sin t, \quad t \geq 0$

32. $x = \sin t, \quad y = \sin 2t$

33. $x = \dfrac{3t}{1 + t^3}, \quad y = \dfrac{3t^2}{1 + t^3}$

34. $x = \cot t, \quad y = 2 \sin^2 t, \quad 0 < t < \pi$

35. If a projectile is fired with an initial speed of v_0 ft/s at an angle α above the horizontal, then its position after t seconds is given by the parametric equations

$$x = (v_0 \cos \alpha)t \qquad y = (v_0 \sin \alpha)t - 16t^2$$

(where x and y are measured in feet). Show that the path of the projectile is a parabola by eliminating the parameter t.

36. Referring to Exercise 35, suppose a gun fires a bullet into the air with an initial speed of 2048 ft/s at an angle of 30° to the horizontal.

(a) After how many seconds will the bullet hit the ground?

(b) How far from the gun will the bullet hit the ground?

(c) What is the maximum height attained by the bullet?

37–42 ■ Use a graphing device to draw the curve represented by the parametric equations.

37. $x = \sin t, \quad y = 2 \cos 3t$

38. $x = 2 \sin t, \quad y = \cos 4t$

39. $x = 3 \sin 5t, \quad y = 5 \cos 3t$

40. $x = \sin 4t, \quad y = \cos 3t$

41. $x = \sin(\cos t), \quad y = \cos(t^{3/2}), \quad 0 \leq t \leq 2\pi$

42. $x = 2 \cos t + \cos 2t, \quad y = 2 \sin t - \sin 2t$

43–46 ■ A polar equation is given.

(a) Express the polar equation in parametric form.

(b) Use a graphing device to graph the parametric equations you found in part (a).

43. $r = 2^{\theta/12}, \quad 0 \leq \theta \leq 4\pi$

44. $r = \sin \theta + 2 \cos \theta$

45. $r = \dfrac{4}{2 - \cos \theta}$

46. $r = 2^{\sin \theta}$

time of radio signals emitted from the satellite. Knowing the distance to three different satellites tells us that we are at the point of intersection of three different spheres. This uniquely determines our position (see Exercise 53, page 643).

Multiplying the second equation by 100 and rewriting the third gives the following system, which we solve using Gaussian elimination.

$$\begin{cases} x + y + z = 50{,}000 \\ 5x + 9y + 16z = 400{,}000 \\ x - 3z = 0 \end{cases}$$ 100 × Equation 2; Subtract 3z

$$\begin{cases} x + y + z = 50{,}000 \\ 4y + 11z = 150{,}000 \\ -y - 4z = -50{,}000 \end{cases}$$ Equation 2 + (−5) × Equation 1 = new Equation 2; Equation 3 + (−1) × Equation 1 = new Equation 3

$$\begin{cases} x + y + z = 50{,}000 \\ -5z = -50{,}000 \\ -y - 4z = -50{,}000 \end{cases}$$ Equation 2 + 4 × Equation 3 = new Equation 2

$$\begin{cases} x + y + z = 50{,}000 \\ z = 10{,}000 \\ y + 4z = 50{,}000 \end{cases}$$ $(-\frac{1}{5})$ × Equation 2; (−1) × Equation 3

$$\begin{cases} x + y + z = 50{,}000 \\ y + 4z = 50{,}000 \\ z = 10{,}000 \end{cases}$$ Interchange Equations 2 and 3

Now that the system is in triangular form, we use back-substitution to find that $x = 30{,}000$, $y = 10{,}000$, and $z = 10{,}000$. This means that John should invest

$30,000 in the money market fund

$10,000 in the blue-chip stock fund

$10,000 in the high-tech stock fund ■

9.3 Exercises

1–4 ■ State whether the equation or system of equations is linear.

1. $6x - \sqrt{3}y + \frac{1}{2}z = 0$

2. $x^2 + y^2 + z^2 = 4$

3. $\begin{cases} xy - 3y + z = 5 \\ x - y^2 + 5z = 0 \\ 2x + yz = 3 \end{cases}$

4. $\begin{cases} x - 2y + 3z = 10 \\ 2x + 5y = 2 \\ y + 2z = 4 \end{cases}$

5–10 ■ Use back-substitution to solve the triangular system.

5. $\begin{cases} x - 2y + 4z = 3 \\ y + 2z = 7 \\ z = 2 \end{cases}$

6. $\begin{cases} x + y - 3z = 8 \\ y - 3z = 5 \\ z = -1 \end{cases}$

7. $\begin{cases} x + 2y + z = 7 \\ -y + 3z = 9 \\ 2z = 6 \end{cases}$

8. $\begin{cases} x - 2y + 3z = 10 \\ 2y - z = 2 \\ 3z = 12 \end{cases}$

9. $\begin{cases} 2x - y + 6z = 5 \\ y + 4z = 0 \\ -2z = 1 \end{cases}$

10. $\begin{cases} 4x + 3z = 10 \\ 2y - z = -6 \\ \frac{1}{2}z = 4 \end{cases}$

11–14 ■ Perform an operation on the given system that eliminates the indicated variable. Write the new equivalent system.

11. $\begin{cases} x - 2y - z = 4 \\ x - y + 3z = 0 \\ 2x + y + z = 0 \end{cases}$ Eliminate the x-term from the second equation.

12. $\begin{cases} x + y - 3z = 3 \\ -2x + 3y + z = 2 \\ x - y + 2z = 0 \end{cases}$ Eliminate the x-term from the second equation.

13. $\begin{cases} 2x - y + 3z = 2 \\ x + 2y - z = 4 \\ -4x + 5y + z = 10 \end{cases}$ Eliminate the x-term from the third equation.

14. $\begin{cases} x - 4y + z = 3 \\ y - 3z = 10 \\ 3y - 8z = 24 \end{cases}$ Eliminate the y-term from the third equation.

15–32 ■ Find the complete solution of the linear system, or show that it is inconsistent.

15. $\begin{cases} x + y + z = 4 \\ x + 3y + 3z = 10 \\ 2x + y - z = 3 \end{cases}$

16. $\begin{cases} x + y + z = 0 \\ -x + 2y + 5z = 3 \\ 3x - y = 6 \end{cases}$

17. $\begin{cases} x - 4z = 1 \\ 2x - y - 6z = 4 \\ 2x + 3y - 2z = 8 \end{cases}$

18. $\begin{cases} x - y + 2z = 2 \\ 3x + y + 5z = 8 \\ 2x - y - 2z = -7 \end{cases}$

19. $\begin{cases} 2x + 4y - z = 2 \\ x + 2y - 3z = -4 \\ 3x - y + z = 1 \end{cases}$

20. $\begin{cases} 2x + y - z = -8 \\ -x + y + z = 3 \\ -2x + 4z = 18 \end{cases}$

21. $\begin{cases} y - 2z = 0 \\ 2x + 3y = 2 \\ -x - 2y + z = -1 \end{cases}$

22. $\begin{cases} 2y + z = 3 \\ 5x + 4y + 3z = -1 \\ x - 3y = -2 \end{cases}$

23. $\begin{cases} x + 2y - z = 1 \\ 2x + 3y - 4z = -3 \\ 3x + 6y - 3z = 4 \end{cases}$

24. $\begin{cases} -x + 2y + 5z = 4 \\ x - 2z = 0 \\ 4x - 2y - 11z = 2 \end{cases}$

25. $\begin{cases} 2x + 3y - z = 1 \\ x + 2y = 3 \\ x + 3y + z = 4 \end{cases}$

26. $\begin{cases} x - 2y - 3z = 5 \\ 2x + y - z = 5 \\ 4x - 3y - 7z = 5 \end{cases}$

27. $\begin{cases} x + y - z = 0 \\ x + 2y - 3z = -3 \\ 2x + 3y - 4z = -3 \end{cases}$

28. $\begin{cases} x - 2y + z = 3 \\ 2x - 5y + 6z = 7 \\ 2x - 3y - 2z = 5 \end{cases}$

29. $\begin{cases} x + 3y - 2z = 0 \\ 2x + 4z = 4 \\ 4x + 6y = 4 \end{cases}$

30. $\begin{cases} 2x + 4y - z = 3 \\ x + 2y + 4z = 6 \\ x + 2y - 2z = 0 \end{cases}$

31. $\begin{cases} x + z + 2w = 6 \\ y - 2z = -3 \\ x + 2y - z = -2 \\ 2x + y + 3z - 2w = 0 \end{cases}$

32. $\begin{cases} x + y + z + w = 0 \\ x + y + 2z + 2w = 0 \\ 2x + 2y + 3z + 4w = 1 \\ 2x + 3y + 4z + 5w = 2 \end{cases}$

```
rref([A])
  [[1 0 0 5 ]
   [0 1 0 2 ]
   [0 0 1 10]]
```

Figure 4

potassium requirement is 500 mg, we get the first equation below. Similar reasoning for the protein and vitamin D requirements leads to the system

$$\begin{cases} 50x + 75y + 10z = 500 & \text{Potassium} \\ 5x + 10y + 3z = 75 & \text{Protein} \\ 90x + 100y + 50z = 1150 & \text{Vitamin D} \end{cases}$$

Dividing the first equation by 5 and the third one by 10 gives the system

$$\begin{cases} 10x + 15y + 2z = 100 \\ 5x + 10y + 3z = 75 \\ 9x + 10y + 5z = 115 \end{cases}$$

We can solve this system using Gaussian elimination, or we can use a graphing calculator to find the reduced row-echelon form of the augmented matrix of the system. Using the `rref` command on the TI-83, we get the output in Figure 4. From the reduced row-echelon form we see that $x = 5$, $y = 2$, $z = 10$. The subject should be fed 5 oz of MiniCal, 2 oz of LiquiFast, and 10 oz of SlimQuick every day. ■

Check Your Answer

$x = 5, y = 2, z = 10$:

$$\begin{cases} 10(5) + 15(2) + 2(10) = 100 \\ 5(5) + 10(2) + 3(10) = 75 \\ 9(5) + 10(2) + 5(10) = 115 \end{cases} \checkmark$$

A more practical application might involve dozens of foods and nutrients rather than just three. Such problems lead to systems with large numbers of variables and equations. Computers or graphing calculators are essential for solving such large systems.

9.4 Exercises

1–6 ■ State the dimension of the matrix.

1. $\begin{bmatrix} 2 & 7 \\ 0 & -1 \\ 5 & -3 \end{bmatrix}$ **2.** $\begin{bmatrix} -1 & 5 & 4 & 0 \\ 0 & 2 & 11 & 3 \end{bmatrix}$ **3.** $\begin{bmatrix} 12 \\ 35 \end{bmatrix}$

4. $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ **5.** $\begin{bmatrix} 1 & 4 & 7 \end{bmatrix}$ **6.** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

7–14 ■ A matrix is given.

(a) Determine whether the matrix is in row-echelon form.

(b) Determine whether the matrix is in reduced row-echelon form.

(c) Write the system of equations for which the given matrix is the augmented matrix.

7. $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 5 \end{bmatrix}$ **8.** $\begin{bmatrix} 1 & 3 & -3 \\ 0 & 1 & 5 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 2 & 8 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **10.** $\begin{bmatrix} 1 & 0 & -7 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 1 \end{bmatrix}$ **12.** $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ **14.** $\begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

15–24 ■ The system of linear equations has a unique solution. Find the solution using Gaussian elimination or Gauss-Jordan elimination.

15. $\begin{cases} x - 2y + z = 1 \\ y + 2z = 5 \\ x + y + 3z = 8 \end{cases}$ **16.** $\begin{cases} x + y + 6z = 3 \\ x + y + 3z = 3 \\ x + 2y + 4z = 7 \end{cases}$

17. $\begin{cases} x + y + z = 2 \\ 2x - 3y + 2z = 4 \\ 4x + y - 3z = 1 \end{cases}$ **18.** $\begin{cases} x + y + z = 4 \\ -x + 2y + 3z = 17 \\ 2x - y = -7 \end{cases}$

19. $\begin{cases} x + 2y - z = -2 \\ x + z = 0 \\ 2x - y - z = -3 \end{cases}$ **20.** $\begin{cases} 2y + z = 4 \\ x + y = 4 \\ 3x + 3y - z = 10 \end{cases}$

21. $\begin{cases} x_1 + 2x_2 - x_3 = 9 \\ 2x_1 - x_3 = -2 \\ 3x_1 + 5x_2 + 2x_3 = 22 \end{cases}$ **22.** $\begin{cases} 2x_1 + x_2 = 7 \\ 2x_1 - x_2 + x_3 = 6 \\ 3x_1 - 2x_2 + 4x_3 = 11 \end{cases}$

23. $\begin{cases} 2x - 3y - z = 13 \\ -x + 2y - 5z = 6 \\ 5x - y - z = 49 \end{cases}$

24. $\begin{cases} 10x + 10y - 20z = 60 \\ 15x + 20y + 30z = -25 \\ -5x + 30y - 10z = 45 \end{cases}$

25–34 ■ Determine whether the system of linear equations is inconsistent or dependent. If it is dependent, find the complete solution.

25. $\begin{cases} x + y + z = 2 \\ y - 3z = 1 \\ 2x + y + 5z = 0 \end{cases}$ **26.** $\begin{cases} x + 3z = 3 \\ 2x + y - 2z = 5 \\ -y + 8z = 8 \end{cases}$

27. $\begin{cases} 2x - 3y - 9z = -5 \\ x + 3z = 2 \\ -3x + y - 4z = -3 \end{cases}$

28. $\begin{cases} x - 2y + 5z = 3 \\ -2x + 6y - 11z = 1 \\ 3x - 16y + 20z = -26 \end{cases}$

29. $\begin{cases} x - y + 3z = 3 \\ 4x - 8y + 32z = 24 \\ 2x - 3y + 11z = 4 \end{cases}$ **30.** $\begin{cases} -2x + 6y - 2z = -12 \\ x - 3y + 2z = 10 \\ -x + 3y + 2z = 6 \end{cases}$

31. $\begin{cases} x + 4y - 2z = -3 \\ 2x - y + 5z = 12 \\ 8x + 5y + 11z = 30 \end{cases}$ **32.** $\begin{cases} 3r + 2s - 3t = 10 \\ r - s - t = -5 \\ r + 4s - t = 20 \end{cases}$

33. $\begin{cases} 2x + y - 2z = 12 \\ -x - \frac{1}{2}y + z = -6 \\ 3x + \frac{3}{2}y - 3z = 18 \end{cases}$ **34.** $\begin{cases} y - 5z = 7 \\ 3x + 2y = 12 \\ 3x + 10z = 80 \end{cases}$

35–46 ■ Solve the system of linear equations.

35. $\begin{cases} 4x - 3y + z = -8 \\ -2x + y - 3z = -4 \\ x - y + 2z = 3 \end{cases}$ **36.** $\begin{cases} 2x - 3y + 5z = 14 \\ 4x - y - 2z = -17 \\ -x - y + z = 3 \end{cases}$

37. $\begin{cases} x + 2y - 3z = -5 \\ -2x - 4y - 6z = 10 \\ 3x + 7y - 2z = -13 \end{cases}$ **38.** $\begin{cases} 3x - y + 2z = -1 \\ 4x - 2y + z = -7 \\ -x + 3y - 2z = -1 \end{cases}$

39. $\begin{cases} -x + 2y + z - 3w = 3 \\ 3x - 4y + z + w = 9 \\ -x - y + z + w = 0 \\ 2x + y + 4z - 2w = 3 \end{cases}$

40. $\begin{cases} x + y - z - w = 6 \\ 2x + z - 3w = 8 \\ x - y + 4w = -10 \\ 3x + 5y - z - w = 20 \end{cases}$

41. $\begin{cases} x + y + 2z - w = -2 \\ 3y + z + 2w = 2 \\ x + y + 3w = 2 \\ -3x + z + 2w = 5 \end{cases}$

42. $\begin{cases} x - 3y + 2z + w = -2 \\ x - 2y - 2w = -10 \\ z + 5w = 15 \\ 3x + 2z + w = -3 \end{cases}$

43. $\begin{cases} x + z + w = 4 \\ y - z = -4 \\ x - 2y + 3z + w = 12 \\ 2x - 2z + 5w = -1 \end{cases}$

44. $\begin{cases} y - z + 2w = 0 \\ 3x + 2y + w = 0 \\ 2x + 4w = 12 \\ -2x - 2z + 5w = 6 \end{cases}$

45. $\begin{cases} x - y + w = 0 \\ 3x - z + 2w = 0 \\ x - 4y + z + 2w = 0 \end{cases}$ **46.** $\begin{cases} 2x - y + 2z + \\ -x + y + 4z - \\ 3x - 2y - z \end{cases}$

Applications

47. Nutrition A doctor recommends that a patient take 50 mg each of niacin, riboflavin, and thiamin daily to alleviate a vitamin deficiency. In his medicine chest a home, the patient finds three brands of vitamin pills. amounts of the relevant vitamins per pill are given in table. How many pills of each type should he take ev day to get 50 mg of each vitamin?

	VitaMax	Vitron	VitaPlu
Niacin (mg)	5	10	15
Riboflavin (mg)	15	20	0
Thiamin (mg)	10	10	10

48. Mixtures A chemist has three acid solutions at var concentrations. The first is 10% acid, the second is 2 and the third is 40%. How many milliliters of each sl he use to make 100 mL of 18% solution, if he has to four times as much of the 10% solution as the 40% solution?

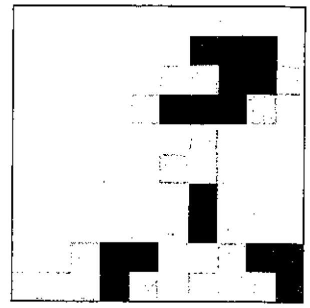
49. Distance, Speed, and Time Amanda, Bryce, and enter a race in which they have to run, swim, and cyc a marked course. Their average speeds are given in tl table. Corey finishes first with a total time of 1 h 45 n Amanda comes in second with a time of 2 h 30 min.

grid that we have used is far too coarse to provide good image resolution. In currently available high-resolution digital cameras use matrices with di 2048 × 2048 or larger.

Once the image is stored as a matrix, it can be manipulated usin operations. For example, to darken the image, we add a constant to each en matrix; to lighten the image, we subtract. To increase the contrast, we d darker areas and lighten the lighter areas, so we could add 1 to each entry th or 6 and subtract 1 from each entry that is 1, 2, or 3. (Note that we cannot c entry of 7 or lighten a 0.) Applying this process to the matrix in Figure 3(c) the new matrix in Figure 4(a). This generates the high-contrast image s Figure 4(b).

$$\begin{bmatrix} 0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&5&7&6&1\\ 0&0&0&0&1&2&2&6&6&2\\ 0&0&0&0&2&6&5&7&2&1\\ 0&0&0&0&0&1&2&1&1&0\\ 0&0&0&0&0&2&2&1&0&0\\ 0&0&0&0&0&0&5&0&0&0\\ 0&0&0&0&1&1&5&1&1&1\\ 1&1&2&6&6&1&1&2&5&5\\ 2&2&2&5&2&1&2&2&2&5 \end{bmatrix}$$

(a) Matrix modified to increase contrast



(b) High-contrast image

Figure 4

Other ways of representing and manipulating images using matrices are d in the *Discovery Projects* on pages 700 and 792.

9.5 Exercises

1–2 ■ Determine whether the matrices A and B are equal.

1. $A = \begin{bmatrix} 1 & -2 & 0 \\ \frac{1}{2} & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ \frac{1}{2} & 6 \end{bmatrix}$

2. $A = \begin{bmatrix} \frac{1}{4} & \ln 1 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0.25 & 0 \\ \sqrt{4} & \frac{6}{2} \end{bmatrix}$

3–10 ■ Perform the matrix operation, or if it is impossible, explain why.

3. $\begin{bmatrix} 2 & 6 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ 6 & 2 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & -2 \end{bmatrix}$

5. $3\begin{bmatrix} 1 & 2 \\ 4 & -1 \\ 1 & 0 \end{bmatrix}$

6. $2\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 6 \\ 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 6 \\ -2 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 1 & 2 \\ 6 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 6 \\ -2 & 0 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 2 & 2 & -1 \end{bmatrix}$

10. $\begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix}$

11–16 ■ Solve the matrix equation for the unknown matrix X, or explain why no solution exists.

$$A = \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 10 & 20 \\ 30 & 20 \\ 10 & 0 \end{bmatrix}$$

11. $2X + A = B$

12. $3X - B = C$

13. $2(B - X) = D$

14. $5(X - C) = D$

15. $\frac{1}{5}(X + D) = C$

16. $2A = B - 3X$

17–38 ■ The matrices A, B, C, D, E, F, and G are defined as follows.

$$A = \begin{bmatrix} 2 & -5 \\ 0 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & \frac{1}{2} & 5 \\ 1 & -1 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & -\frac{5}{2} & 0 \\ 0 & 2 & -3 \end{bmatrix}$$

$$D = \begin{bmatrix} 7 & 3 \end{bmatrix} \qquad E = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad G = \begin{bmatrix} 5 & -3 & 10 \\ 6 & 1 & 0 \\ -5 & 2 & 2 \end{bmatrix}$$

Carry out the indicated algebraic operation, or explain why it cannot be performed.

17. $B + C$

18. $B + F$

19. $C - B$

20. $5A$

21. $3B + 2C$

22. $C - 5A$

23. $2C - 6B$

24. DA

25. AD

26. BC

27. BF

28. GF

29. $(DA)B$

30. $D(AB)$

31. GE

32. A^2

33. A^3

34. $DB + DC$

35. B^2

36. F^2

37. $BF + FE$

38. ABE

39–42 ■ Solve for x and y.

39. $\begin{bmatrix} x & 2y \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2x & -6y \end{bmatrix}$

40. $3\begin{bmatrix} x & y \\ y & x \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ -9 & 6 \end{bmatrix}$

41. $2\begin{bmatrix} x & y \\ x + y & x - y \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -2 & 6 \end{bmatrix}$

42. $\begin{bmatrix} x & y \\ -y & x \end{bmatrix} - \begin{bmatrix} y & x \\ x & -y \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -6 & 6 \end{bmatrix}$

43–46 ■ Write the system of equations as a matrix equation (see Example 6).

43. $\begin{cases} 2x - 5y = 7 \\ 3x + 2y = 4 \end{cases}$

44. $\begin{cases} 6x - y + z = 12 \\ 2x \quad\;\; + z = 7 \\ \quad\;\; y - 2z = 4 \end{cases}$

45. $\begin{cases} 3x_1 + 2x_2 - x_3 + x_4 = 0 \\ x_1 \qquad\; - x_3 \qquad = 5 \\ \qquad 3x_2 + x_3 - x_4 = 4 \end{cases}$

46. $\begin{cases} x - y + z = 2 \\ 4x - 2y - z = 2 \\ x + y + 5z = 2 \\ -x - y - z = 2 \end{cases}$

47. Let

$$A = \begin{bmatrix} 1 & 0 & 6 & -1 \\ 2 & \frac{1}{2} & 4 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 7 & -9 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}$$

Determine which of the following products are defined, and calculate the ones that are:

$ABC \qquad ACB \qquad BAC$

$BCA \qquad CAB \qquad CBA$

> Since Lotka and Volterra's time, more detailed mathematical models of animal populations have been developed. For many species the population is divided into several stages—immature, juvenile, adult, and so on. The proportion of each stage that survives or reproduces in a given time period is entered into a matrix (called a transition matrix); matrix multiplication is then used to predict the population in succeeding time periods. (See the *Discovery Project*, page 688.)
>
> As you can see, the power of mathematics to model and predict is an invaluable tool in the ongoing debate over the environment.

Then we can write these matrix equations as

$$AX = B \qquad \text{Hamster equation}$$
$$AY = C \qquad \text{Gerbil equation}$$

We want to solve for X and Y, so we multiply both sides of each equation by A^{-1}, the inverse of the coefficient matrix. We could find A^{-1} by hand, but it is more convenient to use a graphing calculator as shown in Figure 3.

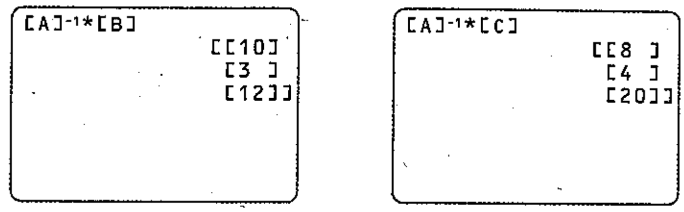


Figure 3 (a) (b)

From the calculator displays, we see that

$$X = A^{-1}B = \begin{bmatrix} 10 \\ 3 \\ 12 \end{bmatrix}, \qquad Y = A^{-1}C = \begin{bmatrix} 8 \\ 4 \\ 20 \end{bmatrix}$$

Thus, each hamster should be fed 10 g of KayDee Food, 3 g of Pet Pellets, and 12 g of Rodent Chow, and each gerbil should be fed 8 g of KayDee Food, 4 g of Pet Pellets, and 20 g of Rodent Chow daily. ■

9.6 Exercises

1–4 ■ Calculate the products AB and BA to verify that B is the inverse of A.

1. $A = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & -3 \\ 4 & -7 \end{bmatrix}$, $B = \begin{bmatrix} \frac{7}{2} & -\frac{3}{2} \\ 2 & -1 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 4 & 0 \\ -1 & -3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 8 & -3 & 4 \\ -2 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$

4. $A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 1 & -6 \\ 2 & 1 & 12 \end{bmatrix}$, $B = \begin{bmatrix} 9 & -10 & -8 \\ -12 & 14 & 11 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

5–6 ■ Find the inverse of the matrix and verify that $A^{-1}A = AA^{-1} = I_2$ and $B^{-1}B = BB^{-1} = I_3$.

5. $A = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}$

6. $B = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 2 \\ -2 & -1 & 0 \end{bmatrix}$

7–22 ■ Find the inverse of the matrix if it exists.

7. $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$

8. $\begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix}$

9. $\begin{bmatrix} 2 & 5 \\ -5 & -13 \end{bmatrix}$

10. $\begin{bmatrix} -7 & 4 \\ 8 & -5 \end{bmatrix}$

11. $\begin{bmatrix} 6 & -3 \\ -8 & 4 \end{bmatrix}$

12. $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 5 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 0.4 & -1.2 \\ 0.3 & 0.6 \end{bmatrix}$

14. $\begin{bmatrix} 4 & 2 & 3 \\ 3 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 2 & 4 & 1 \\ -1 & 1 & -1 \\ 1 & 4 & 0 \end{bmatrix}$

16. $\begin{bmatrix} 5 & 7 & 4 \\ 3 & -1 & 3 \\ 6 & 7 & 5 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & -1 \\ 1 & -1 & -10 \end{bmatrix}$

18. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 4 \\ 2 & 1 & 2 \end{bmatrix}$

19. $\begin{bmatrix} 0 & -2 & 2 \\ 3 & 1 & 3 \\ 1 & -2 & 3 \end{bmatrix}$

20. $\begin{bmatrix} 3 & -2 & 0 \\ 5 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & 2 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

23–30 ■ Solve the system of equations by converting to a matrix equation and using the inverse of the coefficient matrix, as in Example 6. Use the inverses from Exercises 7–10, 15, 16, 19, and 21.

23. $\begin{cases} 5x + 3y = 4 \\ 3x + 2y = 0 \end{cases}$

24. $\begin{cases} 3x + 4y = 10 \\ 7x + 9y = 20 \end{cases}$

25. $\begin{cases} 2x + 5y = 2 \\ -5x - 13y = 20 \end{cases}$

26. $\begin{cases} -7x + 4y = 0 \\ 8x - 5y = 100 \end{cases}$

27. $\begin{cases} 2x + 4y + z = 7 \\ -x + y - z = 0 \\ x + 4y = -2 \end{cases}$

28. $\begin{cases} 5x + 7y + 4z = 1 \\ 3x - y + 3z = 1 \\ 6x + 7y + 5z = 1 \end{cases}$

29. $\begin{cases} -2y + 2z = 12 \\ 3x + y + 3z = -2 \\ x - 2y + 3z = 8 \end{cases}$

30. $\begin{cases} x + 2y + 3w = 0 \\ y + z + w = 1 \\ y + w = 2 \\ x + 2y + 2w = 3 \end{cases}$

31–36 ■ Use a calculator that can perform matrix operatio to solve the system, as in Example 7.

31. $\begin{cases} x + y - 2z = 3 \\ 2x + 5z = 11 \\ 2x + 3y = 12 \end{cases}$

32. $\begin{cases} 3x + 4y - z = 2 \\ 2x - 3y + z = -5 \\ 5x - 2y + 2z = -3 \end{cases}$

33. $\begin{cases} 12x + \frac{1}{2}y - 7z = 21 \\ 11x - 2y + 3z = 43 \\ 13x + y - 4z = 29 \end{cases}$

34. $\begin{cases} x + \frac{1}{2}y - \frac{1}{3}z = 4 \\ x - \frac{1}{4}y + \frac{1}{6}z = 7 \\ x + y - z = -6 \end{cases}$

35. $\begin{cases} x + y - 3w = 0 \\ x - 2z = 8 \\ 2y - z + w = 5 \\ 2x + 3y - 2w = 13 \end{cases}$

36. $\begin{cases} x + y + z + w = 15 \\ x - y + z - w = 5 \\ x + 2y + 3z + 4w = 26 \\ x - 2y + 3z - 4w = 2 \end{cases}$

9.7 Exercises

1–8 ■ Find the determinant of the matrix, if it exists.

1. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ **2.** $\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 4 & 5 \\ 0 & -1 \end{bmatrix}$ **4.** $\begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 5 \end{bmatrix}$ **6.** $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

7. $\begin{bmatrix} \frac{1}{2} & \frac{1}{8} \\ 1 & \frac{1}{2} \end{bmatrix}$ **8.** $\begin{bmatrix} 2.2 & -1.4 \\ 0.5 & 1.0 \end{bmatrix}$

9–14 ■ Evaluate the minor and cofactor using the matrix A.

$$A = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ -3 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

9. M_{11}, A_{11} **10.** M_{33}, A_{33} **11.** M_{12}, A_{12}

12. M_{13}, A_{13} **13.** M_{23}, A_{23} **14.** M_{32}, A_{32}

15–22 ■ Find the determinant of the matrix. Determine whether the matrix has an inverse, but don't calculate the inverse.

15. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 1 & -3 \end{bmatrix}$ **16.** $\begin{bmatrix} 0 & -1 & 0 \\ 2 & 6 & 4 \\ 1 & 0 & 3 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 3 & 7 \\ 2 & 0 & -1 \\ 0 & 2 & 6 \end{bmatrix}$ **18.** $\begin{bmatrix} -2 & -\frac{3}{2} & \frac{1}{2} \\ 2 & 4 & 0 \\ \frac{1}{2} & 2 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 30 & 0 & 20 \\ 0 & -10 & -20 \\ 40 & 0 & 10 \end{bmatrix}$ **20.** $\begin{bmatrix} 1 & 2 & 5 \\ -2 & -3 & 2 \\ 3 & 5 & 3 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 3 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ -1 & 0 & 0 & 2 \\ 1 & 6 & 4 & 1 \end{bmatrix}$ **22.** $\begin{bmatrix} 1 & 2 & 0 & 2 \\ 3 & -4 & 0 & 4 \\ 0 & 1 & 6 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}$

23–26 ■ Evaluate the determinant, using row or column operations whenever possible to simplify your work.

23. $\begin{vmatrix} 0 & 0 & 4 & 6 \\ 2 & 1 & 1 & 3 \\ 2 & 1 & 2 & 3 \\ 3 & 0 & 1 & 7 \end{vmatrix}$ **24.** $\begin{vmatrix} -2 & 3 & -1 & 7 \\ 4 & 6 & -2 & 3 \\ 7 & 7 & 0 & 5 \\ 3 & -12 & 4 & 0 \end{vmatrix}$

25. $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 4 & 6 & 8 \\ 0 & 0 & 3 & 6 & 9 \\ 0 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 0 & 5 \end{vmatrix}$ **26.** $\begin{vmatrix} 2 & -1 & 6 & 4 \\ 7 & 2 & -2 & 5 \\ 4 & -2 & 10 & 8 \\ 6 & 1 & 1 & 4 \end{vmatrix}$

27. Let

$$B = \begin{bmatrix} 4 & 1 & 0 \\ -2 & -1 & 1 \\ 4 & 0 & 3 \end{bmatrix}$$

(a) Evaluate $\det(B)$ by expanding by the second row.

(b) Evaluate $\det(B)$ by expanding by the third column.

(c) Do your results in parts (a) and (b) agree?

28. Consider the system

$$\begin{cases} x + 2y + 6z = 5 \\ -3x - 6y + 5z = 8 \\ 2x + 6y + 9z = 7 \end{cases}$$

(a) Verify that $x = -1, y = 0, z = 1$ is a solution of the system.

(b) Find the determinant of the coefficient matrix.

(c) Without solving the system, determine whether there are any other solutions.

(d) Can Cramer's Rule be used to solve this system? Why or why not?

29–44 ■ Use Cramer's Rule to solve the system.

29. $\begin{cases} 2x - y = -9 \\ x + 2y = 8 \end{cases}$ **30.** $\begin{cases} 6x + 12y = 33 \\ 4x + 7y = 20 \end{cases}$

31. $\begin{cases} x - 6y = 3 \\ 3x + 2y = 1 \end{cases}$ **32.** $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 1 \\ \frac{1}{4}x - \frac{1}{6}y = -\frac{3}{2} \end{cases}$

33. $\begin{cases} 0.4x + 1.2y = 0.4 \\ 1.2x + 1.6y = 3.2 \end{cases}$ **34.** $\begin{cases} 10x - 17y = 21 \\ 20x - 31y = 39 \end{cases}$

35. $\begin{cases} x - y + 2z = 0 \\ 3x + z = 11 \\ -x + 2y = 0 \end{cases}$ **36.** $\begin{cases} 5x - 3y + z = 6 \\ 4y - 6z = 22 \\ 7x + 10y = -13 \end{cases}$

37. $\begin{cases} 2x_1 + 3x_2 - 5x_3 = 1 \\ x_1 + x_2 - x_3 = 2 \\ 2x_2 + x_3 = 8 \end{cases}$ **38.** $\begin{cases} -2a + c = 2 \\ a + 2b - c = 9 \\ 3a + 5b + 2c = 22 \end{cases}$

39. $\begin{cases} \frac{1}{3}x - \frac{1}{5}y + \frac{1}{2}z = \frac{7}{10} \\ -\frac{2}{3}x + \frac{2}{5}y + \frac{3}{2}z = \frac{11}{10} \\ x - \frac{4}{5}y + z = \frac{9}{5} \end{cases}$ **40.** $\begin{cases} 2x - y = 5 \\ 5x + 3z = 19 \\ 4y + 7z = 17 \end{cases}$

41. $\begin{cases} 3y + 5z = 4 \\ 2x - z = 10 \\ 4x + 7y = 0 \end{cases}$ **42.** $\begin{cases} 2x - 5y = 4 \\ x + y - z = 8 \\ 3x + 5z = 0 \end{cases}$

Example 5 Using Long Division to Prepare for Partial Fractions

Find the partial fraction decomposition of

$$\frac{2x^4 + 4x^3 - 2x^2 + x + 7}{x^3 + 2x^2 - x - 2}$$

Solution Since the degree of the numerator is larger than the degree of the denominator, we use long division to obtain

$$\begin{array}{r} 2x \\ x^3 + 2x^2 - x - 2 \overline{)2x^4 + 4x^3 - 2x^2 + x + 7} \\ \underline{2x^4 + 4x^3 - 2x^2 - 4x} \\ 5x + 7 \end{array}$$

$$\frac{2x^4 + 4x^3 - 2x^2 + x + 7}{x^3 + 2x^2 - x - 2} = 2x + \frac{5x + 7}{x^3 + 2x^2 - x - 2}$$

The remainder term now satisfies the requirement that the degree of the nume is less than the degree of the denominator. At this point we proceed as in Exa to obtain the decomposition

$$\frac{2x^4 + 4x^3 - 2x^2 + x + 7}{x^3 + 2x^2 - x - 2} = 2x + \frac{2}{x - 1} + \frac{-1}{x + 1} + \frac{-1}{x + 2}$$

9.8 Exercises

1–10 ■ Write the form of the partial fraction decomposition of the function (as in Example 4). Do not determine the numerical values of the coefficients.

1. $\frac{1}{(x-1)(x+2)}$
2. $\frac{x}{x^2+3x-4}$
3. $\frac{x^2-3x+5}{(x-2)^2(x+4)}$
4. $\frac{1}{x^4-x^3}$
5. $\frac{x^2}{(x-3)(x^2+4)}$
6. $\frac{1}{x^4-1}$
7. $\frac{x^3-4x^2+2}{(x^2+1)(x^2+2)}$
8. $\frac{x^4+x^2+1}{x^2(x^2+4)^2}$
9. $\frac{x^3+x+1}{x(2x-5)^3(x^2+2x+5)^2}$
10. $\frac{1}{(x^3-1)(x^2-1)}$

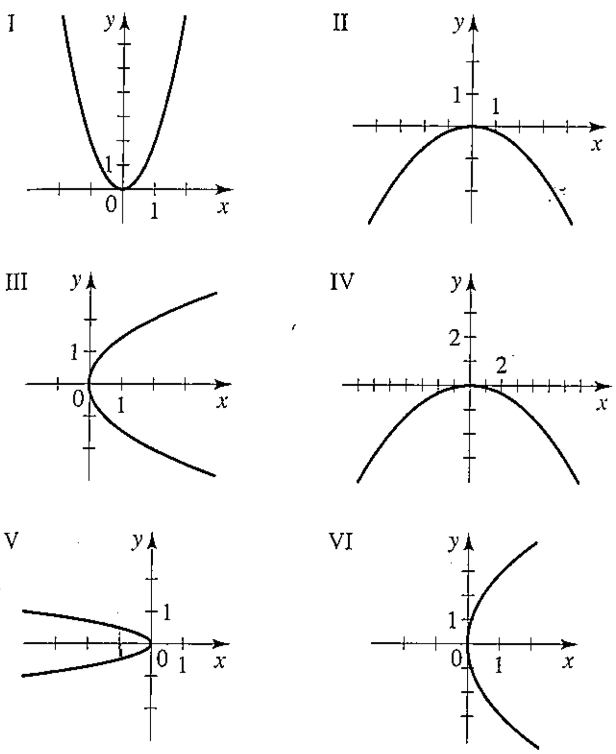
11–42 ■ Find the partial fraction decomposition of the rational function.

11. $\frac{2}{(x-1)(x+1)}$
12. $\frac{2x}{(x-1)(x+1)}$
13. $\frac{5}{(x-1)(x+4)}$
14. $\frac{x+6}{x(x+3)}$
15. $\frac{12}{x^2-9}$
16. $\frac{x-12}{x^2-4x}$
17. $\frac{4}{x^2-4}$
18. $\frac{2x+1}{x^2+x-2}$
19. $\frac{x+14}{x^2-2x-8}$
20. $\frac{8x-3}{2x^2-x}$
21. $\frac{x}{8x^2-10x+3}$
22. $\frac{7x-3}{x^3+2x^2-3x}$
23. $\frac{9x^2-9x+6}{2x^3-x^2-8x+4}$
24. $\frac{-3x^2-3x+2}{(x+2)(2x^2+3x-}$
25. $\frac{x^2+1}{x^3+x^2}$
26. $\frac{3x^2+5x-13}{(3x+2)(x^2-4x+}$
27. $\frac{2x}{4x^2+12x+9}$
28. $\frac{x-4}{(2x-5)^2}$
29. $\frac{4x^2-x-2}{x^4+2x^3}$
30. $\frac{x^3-2x^2-4x+3}{x^4}$
31. $\frac{-10x^2+27x-14}{(x-1)^3(x+2)}$
32. $\frac{-2x^2+5x-1}{x^4-2x^3+2x-1}$
33. $\frac{3x^3+22x^2+53x+41}{(x+2)^2(x+3)^2}$
34. $\frac{3x^2+12x-20}{x^4-8x^2+16}$
35. $\frac{x-3}{x^3+3x}$
36. $\frac{3x^2-2x+8}{x^3-x^2+2x-2}$
37. $\frac{2x^3+7x+5}{(x^2+x+2)(x^2+1)}$
38. $\frac{x^2+x+1}{2x^4+3x^2+1}$

10.1 Exercises

1–6 ■ Match the equation with the graphs labeled I–VI. Give reasons for your answers.

1. $y^2 = 2x$
2. $y^2 = -\frac{1}{4}x$
3. $x^2 = -6y$
4. $2x^2 = y$
5. $y^2 - 8x = 0$
6. $12y + x^2 = 0$



7–18 ■ Find the focus, directrix, and focal diameter of the parabola, and sketch its graph.

7. $y^2 = 4x$
8. $x^2 = y$
9. $x^2 = 9y$
10. $y^2 = 3x$
11. $y = 5x^2$
12. $y = -2x^2$
13. $x = -8y^2$
14. $x = \frac{1}{2}y^2$
15. $x^2 + 6y = 0$
16. $x - 7y^2 = 0$
17. $5x + 3y^2 = 0$
18. $8x^2 + 12y = 0$

19–24 ■ Use a graphing device to graph the parabola.

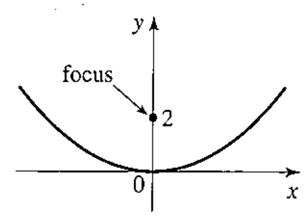
19. $x^2 = 16y$
20. $x^2 = -8y$
21. $y^2 = -\frac{1}{3}x$
22. $8y^2 = x$
23. $4x + y^2 = 0$
24. $x - 2y^2 = 0$

25–36 ■ Find an equation for the parabola that has its vertex at the origin and satisfies the given condition(s).

25. Focus $F(0, 2)$
26. Focus $F(0, -\frac{1}{2})$
27. Focus $F(-8, 0)$
28. Focus $F(5, 0)$
29. Directrix $x = 2$
30. Directrix $y = 6$
31. Directrix $y = -10$
32. Directrix $x = -\frac{1}{8}$
33. Focus on the positive x-axis, 2 units away from the directrix
34. Directrix has y-intercept 6
35. Opens upward with focus 5 units from the vertex
36. Focal diameter 8 and focus on the negative y-axis

37–46 ■ Find an equation of the parabola whose graph is shown.

37.



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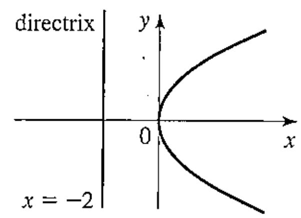

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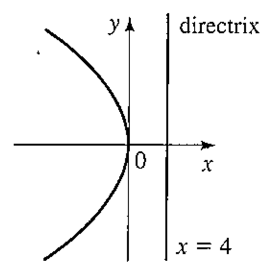

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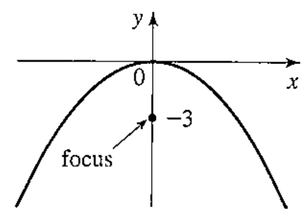

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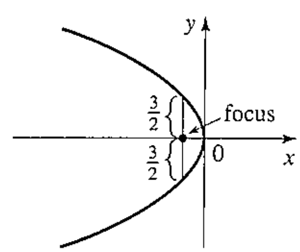

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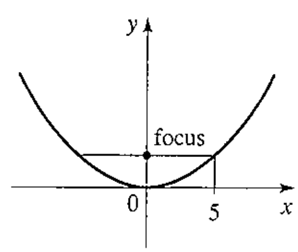

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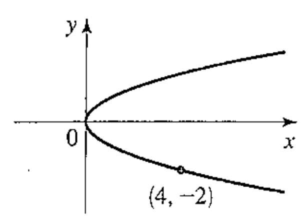

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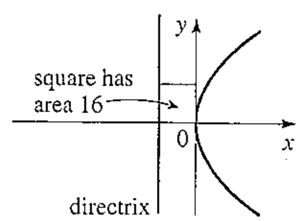


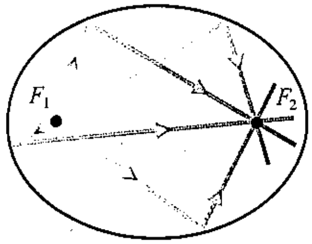


Figure 10

Ellipses, like parabolas, have an interesting *reflection property* that leads to a number of practical applications. If a light source is placed at one focus of a reflecting surface with elliptical cross sections, then all the light will be reflected off the surface to the other focus, as shown in Figure 10. This principle, which works for sound waves as well as for light, is used in *lithotripsy*, a treatment for kidney stones. The patient is placed in a tub of water with elliptical cross sections in such a way that the kidney stone is accurately located at one focus. High-intensity sound waves generated at the other focus are reflected to the stone and destroy it with minimal damage to surrounding tissue. The patient is spared the trauma of surgery and recovers within days instead of weeks.

The reflection property of ellipses is also used in the construction of *whispering galleries*. Sound coming from one focus bounces off the walls and ceiling of an elliptical room and passes through the other focus. In these rooms even quiet whispers spoken at one focus can be heard clearly at the other. Famous whispering galleries include the National Statuary Hall of the U.S. Capitol in Washington, D.C. (see page 771), and the Mormon Tabernacle in Salt Lake City, Utah.

10.2 Exercises

1–4 ■ Match the equation with the graphs labeled I–IV. Give reasons for your answers.

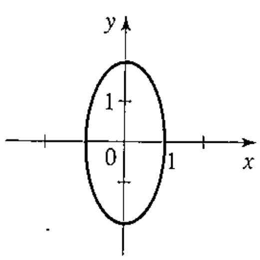
1. $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$

2. $x^2 + \dfrac{y^2}{9} = 1$

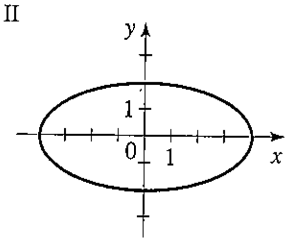
3. $4x^2 + y^2 = 4$

4. $16x^2 + 25y^2 = 400$

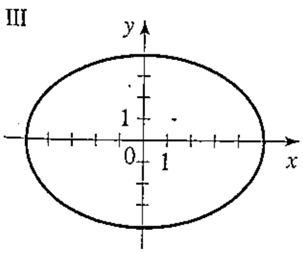
I



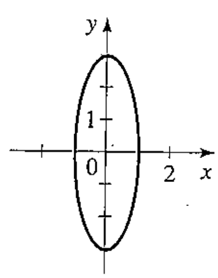
II



III



IV



5–18 ■ Find the vertices, foci, and eccentricity of the ellipse. Determine the lengths of the major and minor axes, and sketch the graph.

5. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

6. $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$

7. $9x^2 + 4y^2 = 36$

8. $4x^2 + 25y^2 = 100$

9. $x^2 + 4y^2 = 16$

10. $4x^2 + y^2 = 16$

11. $2x^2 + y^2 = 3$

12. $5x^2 + 6y^2 = 30$

13. $x^2 + 4y^2 = 1$

14. $9x^2 + 4y^2 = 1$

15. $\frac{1}{2}x^2 + \frac{1}{8}y^2 = \frac{1}{4}$

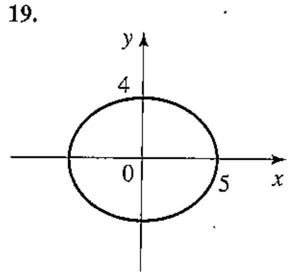
16. $x^2 = 4 - 2y^2$

17. $y^2 = 1 - 2x^2$

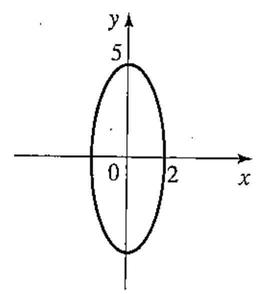
18. $20x^2 + 4y^2 = 5$

19–24 ■ Find an equation for the ellipse whose graph is shown.

19.



20.



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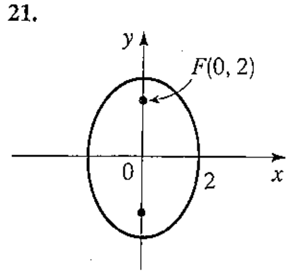

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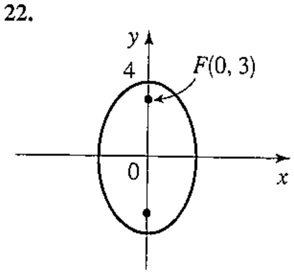

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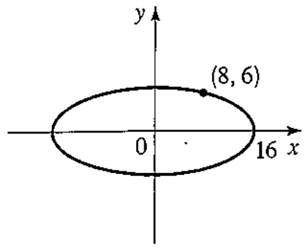

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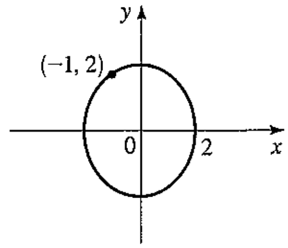


25–28 ■ Use a graphing device to graph the ellipse.

25. $\frac{x^2}{25} + \frac{y^2}{20} = 1$ **26.** $x^2 + \frac{y^2}{12} = 1$

27. $6x^2 + y^2 = 36$ **28.** $x^2 + 2y^2 = 8$

29–40 ■ Find an equation for the ellipse that satisfies the given conditions.

29. Foci $(\pm 4, 0)$, vertices $(\pm 5, 0)$

30. Foci $(0, \pm 3)$, vertices $(0, \pm 5)$

31. Length of major axis 4, length of minor axis 2, foci on y-axis

32. Length of major axis 6, length of minor axis 4, foci on x-axis

33. Foci $(0, \pm 2)$, length of minor axis 6

34. Foci $(\pm 5, 0)$, length of major axis 12

35. Endpoints of major axis $(\pm 10, 0)$, distance between foci 6

36. Endpoints of minor axis $(0, \pm 3)$, distance between foci 8

37. Length of major axis 10, foci on x-axis, ellipse passes through the point $(\sqrt{5}, 2)$

38. Eccentricity $\frac{1}{9}$, foci $(0, \pm 2)$

39. Eccentricity 0.8, foci $(\pm 1.5, 0)$

40. Eccentricity $\sqrt{3}/2$, foci on y-axis, length of major axis 4

41–43 ■ Find the intersection points of the pair of ellipses. Sketch the graphs of each pair of equations on the same coordinate axes and label the points of intersection.

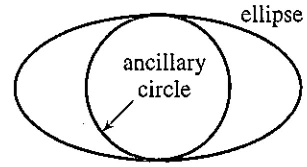
41. $\begin{cases} 4x^2 + y^2 = 4 \\ 4x^2 + 9y^2 = 36 \end{cases}$ **42.** $\begin{cases} \frac{x^2}{16} + \frac{y^2}{9} = 1 \\ \frac{x^2}{9} + \frac{y^2}{16} = 1 \end{cases}$

43. $\begin{cases} 100x^2 + 25y^2 = 100 \\ x^2 + \frac{y^2}{9} = 1 \end{cases}$

44. The **ancillary circle** of an ellipse is the circle with radius equal to half the length of the minor axis and center the same as the ellipse (see the figure). The ancillary circ thus the largest circle that can fit within an ellipse.

(a) Find an equation for the ancillary circle of the el $x^2 + 4y^2 = 16$.

(b) For the ellipse and ancillary circle of part (a), sh if (s, t) is a point on the ancillary circle, then (2s point on the ellipse.



45. (a) Use a graphing device to sketch the top half (the in the first and second quadrants) of the family o lipses $x^2 + ky^2 = 100$ for $k = 4, 10, 25$, and 50.

(b) What do the members of this family of ellipses h common? How do they differ?

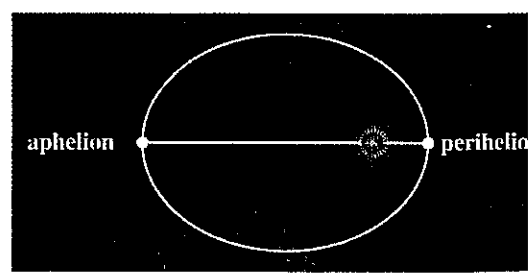
46. If $k > 0$, the following equation represents an ellipse

$$\frac{x^2}{k} + \frac{y^2}{4 + k} = 1$$

Show that all the ellipses represented by this equatio the same foci, no matter what the value of k.

Applications

47. Perihelion and Aphelion The planets move aroun sun in elliptical orbits with the sun at one focus. The in the orbit at which the planet is closest to the sun is **perihelion**, and the point at which it is farthest is call **aphelion**. These points are the vertices of the orbit. T earth's distance from the sun is 147,000,000 km at pe lion and 153,000,000 km at aphelion. Find an equatic the earth's orbit. (Place the origin at the center of the with the sun on the x-axis.)



48. The Orbit of Pluto With an eccentricity of 0.25, Pluto's orbit is the most eccentric in the solar system. length of the minor axis of its orbit is approximately 10,000,000,000 km. Find the distance between Pluto the sun at perihelion and at aphelion. (See Exercise 4

The LORAN (LOng RAnge Navigation) system was used until the early 19
has now been superseded by the GPS system (see page 656). In the LORAN s
hyperbolas are used onboard a ship to determine its location. In Figure 9 rad
tions at A and B transmit signals simultaneously for reception by the ship at
onboard computer converts the time difference in reception of these signals int
tance difference $d(P, A) - d(P, B)$. From the definition of a hyperbola this
the ship on one branch of a hyperbola with foci at A and B (sketched in black
figure). The same procedure is carried out with two other radio stations at C
and this locates the ship on a second hyperbola (shown in red in the figure). (I
tice, only three stations are needed because one station can be used as a focus f
hyperbolas.) The coordinates of the intersection point of these two hyperbolas,
can be calculated precisely by the computer, give the location of P.

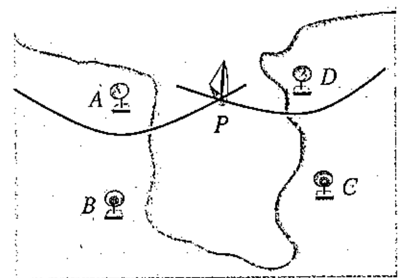


Figure 9
LORAN system for finding the location of a ship

10.3 Exercises

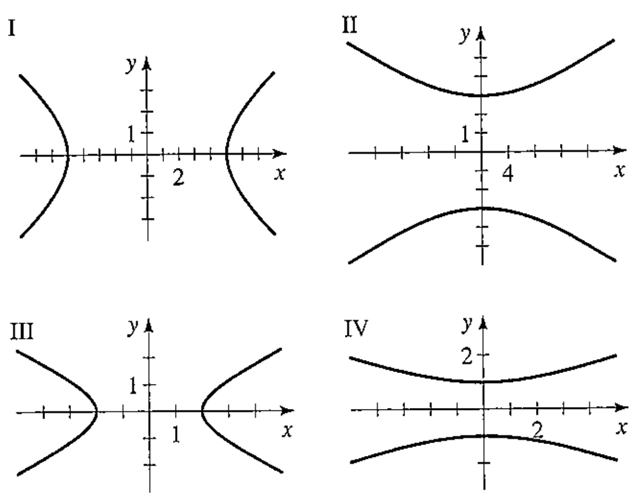
1–4 ■ Match the equation with the graphs labeled I–IV. Give reasons for your answers.

1. $\dfrac{x^2}{4} - y^2 = 1$

2. $y^2 - \dfrac{x^2}{9} = 1$

3. $16y^2 - x^2 = 144$

4. $9x^2 - 25y^2 = 225$



5–16 ■ Find the vertices, foci, and asymptotes of the hy
and sketch its graph.

5. $\dfrac{x^2}{4} - \dfrac{y^2}{16} = 1$

6. $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$

7. $y^2 - \dfrac{x^2}{25} = 1$

8. $\dfrac{x^2}{2} - y^2 = 1$

9. $x^2 - y^2 = 1$

10. $9x^2 - 4y^2 = 36$

11. $25y^2 - 9x^2 = 225$

12. $x^2 - y^2 + 4 = 0$

13. $x^2 - 4y^2 - 8 = 0$

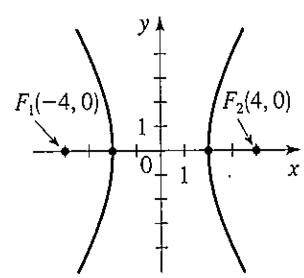
14. $x^2 - 2y^2 = 3$

15. $4y^2 - x^2 = 1$

16. $9x^2 - 16y^2 = 1$

17–22 ■ Find the equation for the hyperbola whose gra
shown.

17.



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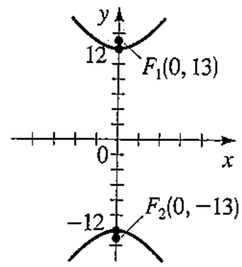

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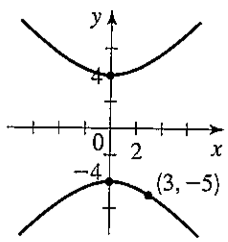

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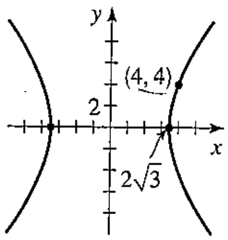

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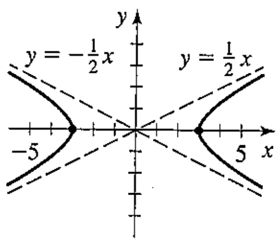

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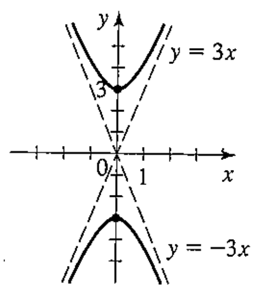


23–26 ■ Use a graphing device to graph the hyperbola.

23. $x^2 - 2y^2 = 8$

24. $3y^2 - 4x^2 = 24$

25. $\dfrac{y^2}{2} - \dfrac{x^2}{6} = 1$

26. $\dfrac{x^2}{100} - \dfrac{y^2}{64} = 1$

27–38 ■ Find an equation for the hyperbola that satisfies the given conditions.

27. Foci $(\pm 5, 0)$, vertices $(\pm 3, 0)$

28. Foci $(0, \pm 10)$, vertices $(0, \pm 8)$

29. Foci $(0, \pm 2)$, vertices $(0, \pm 1)$

30. Foci $(\pm 6, 0)$, vertices $(\pm 2, 0)$

31. Vertices $(\pm 1, 0)$, asymptotes $y = \pm 5x$

32. Vertices $(0, \pm 6)$, asymptotes $y = \pm\frac{1}{3}x$

33. Foci $(0, \pm 8)$, asymptotes $y = \pm\frac{1}{2}x$

34. Vertices $(0, \pm 6)$, hyperbola passes through $(-5, 9)$

35. Asymptotes $y = \pm x$, hyperbola passes through $(5, 3)$

36. Foci $(\pm 3, 0)$, hyperbola passes through $(4, 1)$

37. Foci $(\pm 5, 0)$, length of transverse axis 6

38. Foci $(0, \pm 1)$, length of transverse axis 1

39. **(a)** Show that the asymptotes of the hyperbola $x^2 - y^2 = 5$ are perpendicular to each other.

(b) Find an equation for the hyperbola with foci $(\pm c, 0)$ and with asymptotes perpendicular to each other.

40. The hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{and} \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

are said to be **conjugate** to each other.

(a) Show that the hyperbolas

$$x^2 - 4y^2 + 16 = 0 \qquad \text{and} \qquad 4y^2 - x^2 + 16 = 0$$

are conjugate to each other, and sketch their graphs on the same coordinate axes.

(b) What do the hyperbolas of part (a) have in common?

(c) Show that any pair of conjugate hyperbolas have the relationship you discovered in part (b).

41. In the derivation of the equation of the hyperbola at the beginning of this section, we said that the equation

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = \pm 2a$$

simplifies to

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2)$$

Supply the steps needed to show this.

42. **(a)** For the hyperbola

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

determine the values of a, b, and c, and find the coordinates of the foci F_1 and F_2.

(b) Show that the point $P(5, \frac{16}{3})$ lies on this hyperbola.

(c) Find $d(P, F_1)$ and $d(P, F_2)$.

(d) Verify that the difference between $d(P, F_1)$ and $d(P, F_2)$ is $2a$.

43. Hyperbolas are called **confocal** if they have the same foci.

(a) Show that the hyperbolas

$$\frac{y^2}{k} - \frac{x^2}{16 - k} = 1 \qquad \text{with } 0 < k < 16$$

are confocal.

whether this is in fact the case, we complete the squares:

$$9(x^2 + 2x \quad) - (y^2 - 6y \quad) = 0 \qquad \text{Group terms and factor 9}$$

$$9(x^2 + 2x + 1) - (y^2 - 6y + 9) = 0 + 9 \cdot 1 - 9 \qquad \text{Complete the square}$$

$$9(x + 1)^2 - (y - 3)^2 = 0 \qquad \text{Factor}$$

$$(x + 1)^2 - \frac{(y - 3)^2}{9} = 0 \qquad \text{Divide by 9}$$

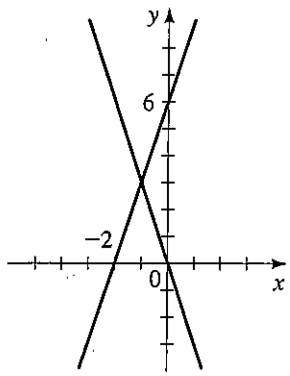


Figure 7
$9x^2 - y^2 + 18x + 6y = 0$

For this to fit the form of the equation of a hyperbola, we would need a nonzero constant to the right of the equal sign. In fact, further analysis shows that this is the equation of a pair of intersecting lines:

$$(y - 3)^2 = 9(x + 1)^2$$

$$y - 3 = \pm 3(x + 1) \qquad \text{Take square roots}$$

$$y = 3(x + 1) + 3 \quad \text{or} \quad y = -3(x + 1) + 3$$

$$y = 3x + 6 \qquad\qquad y = -3x$$

These lines are graphed in Figure 7. ■

Because the equation in Example 4 looked at first glance like the equation of a hyperbola but, in fact, turned out to represent simply a pair of lines, we refer to its graph as a **degenerate hyperbola**. Degenerate ellipses and parabolas can also arise when we complete the square(s) in an equation that seems to represent a conic. For example, the equation

$$4x^2 + y^2 - 8x + 2y + 6 = 0$$

looks as if it should represent an ellipse, because the coefficients of x^2 and y^2 have the same sign. But completing the squares leads to

$$(x - 1)^2 + \frac{(y + 1)^2}{4} = -\frac{1}{4}$$

which has no solution at all (since the sum of two squares cannot be negative). This equation is therefore degenerate.

10.4 Exercises

1–4 ■ Find the center, foci, and vertices of the ellipse, and determine the lengths of the major and minor axes. Then sketch the graph.

1. $\dfrac{(x - 2)^2}{9} + \dfrac{(y - 1)^2}{4} = 1$ **2.** $\dfrac{(x - 3)^2}{16} + (y + 3)^2 = 1$

3. $\dfrac{x^2}{9} + \dfrac{(y + 5)^2}{25} = 1$ **4.** $\dfrac{(x + 2)^2}{4} + y^2 = 1$

5–8 ■ Find the vertex, focus, and directrix of the parabola, and sketch the graph.

5. $(x - 3)^2 = 8(y + 1)$ **6.** $(y + 5)^2 = -6x + 12$

7. $-4(x + \frac{1}{2})^2 = y$ **8.** $y^2 = 16x - 8$

9–12 ■ Find the center, foci, vertices, and asymptotes of the hyperbola. Then sketch the graph.

9. $\dfrac{(x + 1)^2}{9} - \dfrac{(y - 3)^2}{16} = 1$

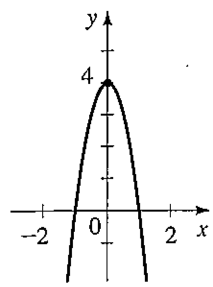
10. $(x - 8)^2 - (y + 6)^2 = 1$

11. $y^2 - \dfrac{(x + 1)^2}{4} = 1$

12. $\dfrac{(y - 1)^2}{25} - (x + 3)^2 = 1$

13–18 ■ Find an equation for the conic whose graph is shown.

13.



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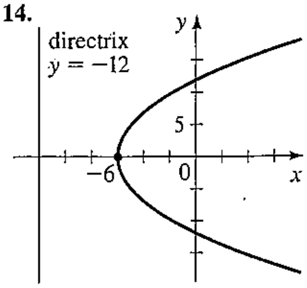

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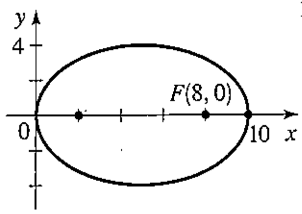

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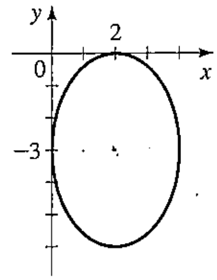

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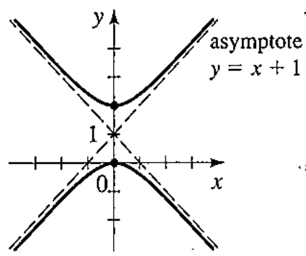

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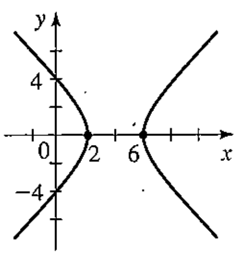


19–30 ■ Complete the square to determine whether the equation represents an ellipse, a parabola, a hyperbola, or a degenerate conic. If the graph is an ellipse, find the center, foci, vertices, and lengths of the major and minor axes. If it is a parabola, find the vertex, focus, and directrix. If it is a hyperbola, find the center, foci, vertices, and asymptotes. Then sketch the graph of the equation. If the equation has no graph, explain why.

19. $9x^2 - 36x + 4y^2 = 0$ **20.** $y^2 = 4(x + 2y)$

21. $x^2 - 4y^2 - 2x + 16y = 20$

22. $x^2 + 6x + 12y + 9 = 0$

23. $4x^2 + 25y^2 - 24x + 250y + 561 = 0$

24. $2x^2 + y^2 = 2y + 1$

25. $16x^2 - 9y^2 - 96x + 288 = 0$

26. $4x^2 - 4x - 8y + 9 = 0$

27. $x^2 + 16 = 4(y^2 + 2x)$ **28.** $x^2 - y^2 = 10(x - y) + 1$

29. $3x^2 + 4y^2 - 6x - 24y + 39 = 0$

30. $x^2 + 4y^2 + 20x - 40y + 300 = 0$

31–34 ■ Use a graphing device to graph the conic.

31. $2x^2 - 4x + y + 5 = 0$

32. $4x^2 + 9y^2 - 36y = 0$

33. $9x^2 + 36 = y^2 + 36x + 6y$

34. $x^2 - 4y^2 + 4x + 8y = 0$

35. Determine what the value of F must be if the graph of equation

$$4x^2 + y^2 + 4(x - 2y) + F = 0$$

is (a) an ellipse, (b) a single point, or (c) the empty se

36. Find an equation for the ellipse that shares a vertex an focus with the parabola $x^2 + y = 100$ and has its othe focus at the origin.

37. This exercise deals with **confocal parabolas**, that is, families of parabolas that have the same focus.

(a) Draw graphs of the family of parabolas

$$x^2 = 4p(y + p)$$

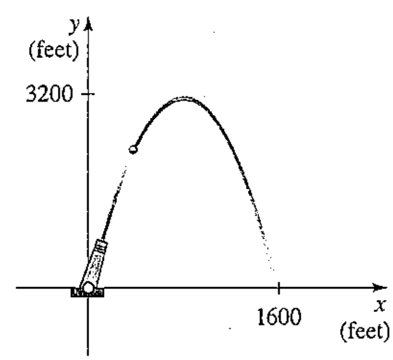
for $p = -2, -\frac{3}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}, 2$.

(b) Show that each parabola in this family has its foc the origin.

(c) Describe the effect on the graph of moving the ve closer to the origin.

Applications

38. Path of a Cannonball A cannon fires a cannonball shown in the figure. The path of the cannonball is a pa with vertex at the highest point of the path. If the cann ball lands 1600 ft from the cannon and the highest poi reaches is 3200 ft above the ground, find an equation f the path of the cannonball. Place the origin at the locat of the cannon.



39. Orbit of a Satellite A satellite is in an elliptical orb around the earth with the center of the earth at one foc The height of the satellite above the earth varies betwe 140 mi and 440 mi. Assume the earth is a sphere with

The following properties of sums are natural consequences of propertie real numbers.

Properties of Sums

Let $a_1, a_2, a_3, a_4, \ldots$ and $b_1, b_2, b_3, b_4, \ldots$ be sequences. Then for every p tive integer n and any real number c, the following properties hold.

1. $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
2. $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$
3. $\sum_{k=1}^{n} ca_k = c\left(\sum_{k=1}^{n} a_k\right)$

Proof To prove Property 1, we write out the left side of the equation t

$$\sum_{k=1}^{n} (a_k + b_k) = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdots + (a_n +$$

Because addition is commutative and associative, we can rearrange the term right side to read

$$\sum_{k=1}^{n} (a_k + b_k) = (a_1 + a_2 + a_3 + \cdots + a_n) + (b_1 + b_2 + b_3 + \cdots +$$

Rewriting the right side using sigma notation gives Property 1. Property 2 is in a similar manner. To prove Property 3, we use the Distributive Property:

$$\sum_{k=1}^{n} ca_k = ca_1 + ca_2 + ca_3 + \cdots + ca_n$$

$$= c(a_1 + a_2 + a_3 + \cdots + a_n) = c\left(\sum_{k=1}^{n} a_k\right)$$

11.1 Exercises

1–10 ■ Find the first four terms and the 100th term of the sequence.

1. $a_n = n + 1$
2. $a_n = 2n + 3$
3. $a_n = \dfrac{1}{n+1}$
4. $a_n = n^2 + 1$
5. $a_n = \dfrac{(-1)^n}{n^2}$
6. $a_n = \dfrac{1}{n^2}$
7. $a_n = 1 + (-1)^n$
8. $a_n = (-1)^{n+1}\dfrac{n}{n+1}$
9. $a_n = n^n$
10. $a_n = 3$

11–16 ■ Find the first five terms of the given recursive defined sequence.

11. $a_n = 2(a_{n-1} - 2)$ and $a_1 = 3$
12. $a_n = \dfrac{a_{n-1}}{2}$ and $a_1 = -8$
13. $a_n = 2a_{n-1} + 1$ and $a_1 = 1$
14. $a_n = \dfrac{1}{1 + a_{n-1}}$ and $a_1 = 1$
15. $a_n = a_{n-1} + a_{n-2}$ and $a_1 = 1, a_2 = 2$
16. $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ and $a_1 = a_2 = a_3 = 1$

17–22 ■ Use a graphing calculator to do the following.

(a) Find the first 10 terms of the sequence.

(b) Graph the first 10 terms of the sequence.

17. $a_n = 4n + 3$

18. $a_n = n^2 + n$

19. $a_n = \dfrac{12}{n}$

20. $a_n = 4 - 2(-1)^n$

21. $a_n = \dfrac{1}{a_{n-1}}$ and $a_1 = 2$

22. $a_n = a_{n-1} - a_{n-2}$ and $a_1 = 1, a_2 = 3$

23–30 ■ Find the nth term of a sequence whose first several terms are given.

23. $2, 4, 8, 16, \ldots$

24. $-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \ldots$

25. $1, 4, 7, 10, \ldots$

26. $5, -25, 125, -625, \ldots$

27. $1, \frac{3}{4}, \frac{5}{9}, \frac{7}{16}, \frac{9}{25}, \ldots$

28. $\frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots$

29. $0, 2, 0, 2, 0, 2, \ldots$

30. $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \ldots$

31–34 ■ Find the first six partial sums $S_1, S_2, S_3, S_4, S_5, S_6$ of the sequence.

31. $1, 3, 5, 7, \ldots$

32. $1^2, 2^2, 3^2, 4^2, \ldots$

33. $\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \frac{1}{3^4}, \ldots$

34. $-1, 1, -1, 1, \ldots$

35–38 ■ Find the first four partial sums and the nth partial sum of the sequence a_n.

35. $a_n = \dfrac{2}{3^n}$

36. $a_n = \dfrac{1}{n+1} - \dfrac{1}{n+2}$

37. $a_n = \sqrt{n} - \sqrt{n+1}$

38. $a_n = \log\left(\dfrac{n}{n+1}\right)$ [*Hint:* Use a property of logarithms to write the nth term as a difference.]

39–46 ■ Find the sum.

39. $\sum_{k=1}^{4} k$

40. $\sum_{k=1}^{4} k^2$

41. $\sum_{k=1}^{3} \frac{1}{k}$

42. $\sum_{j=1}^{100} (-1)^j$

43. $\sum_{i=1}^{8} [1 + (-1)^i]$

44. $\sum_{i=4}^{12} 10$

45. $\sum_{k=1}^{5} 2^{k-1}$

46. $\sum_{i=1}^{3} i2^i$

47–52 ■ Use a graphing calculator to evaluate the sum.

47. $\sum_{k=1}^{10} k^2$

48. $\sum_{k=1}^{100} (3k + 4)$

49. $\sum_{j=7}^{20} j^2(1 + j)$

50. $\sum_{j=5}^{15} \frac{1}{j^2 + 1}$

51. $\sum_{n=0}^{22} (-1)^n 2n$

52. $\sum_{n=1}^{100} \frac{(-1)^n}{n}$

53–58 ■ Write the sum without using sigma notation.

53. $\sum_{k=1}^{5} \sqrt{k}$

54. $\sum_{i=0}^{4} \frac{2i - 1}{2i + 1}$

55. $\sum_{k=0}^{6} \sqrt{k + 4}$

56. $\sum_{k=6}^{9} k(k + 3)$

57. $\sum_{k=3}^{100} x^k$

58. $\sum_{j=1}^{n} (-1)^{j+1} x^j$

59–66 ■ Write the sum using sigma notation.

59. $1 + 2 + 3 + 4 + \cdots + 100$

60. $2 + 4 + 6 + \cdots + 20$

61. $1^2 + 2^2 + 3^2 + \cdots + 10^2$

62. $\dfrac{1}{2 \ln 2} - \dfrac{1}{3 \ln 3} + \dfrac{1}{4 \ln 4} - \dfrac{1}{5 \ln 5} + \cdots + \dfrac{1}{100 \ln 100}$

63. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{999 \cdot 1000}$

64. $\dfrac{\sqrt{1}}{1^2} + \dfrac{\sqrt{2}}{2^2} + \dfrac{\sqrt{3}}{3^2} + \cdots + \dfrac{\sqrt{n}}{n^2}$

65. $1 + x + x^2 + x^3 + \cdots + x^{100}$

66. $1 - 2x + 3x^2 - 4x^3 + 5x^4 + \cdots - 100x^{99}$

67. Find a formula for the nth term of the sequence

$$\sqrt{2},\quad \sqrt{2\sqrt{2}},\quad \sqrt{2\sqrt{2\sqrt{2}}},\quad \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \ldots$$

[*Hint:* Write each term as a power of 2.]

68. Define the sequence

$$G_n = \frac{1}{\sqrt{5}}\left(\frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n}\right)$$

Use the TABLE command on a graphing calculator to find the first 10 terms of this sequence. Compare to the Fibonacci sequence F_n.

Applications

69. Compound Interest Julio deposits $2000 in a savings account that pays 2.4% interest per year compounded

11.2 Exercises

1–4 ■ A sequence is given.
(a) Find the first five terms of the sequence.
(b) What is the common difference d?
(c) Graph the terms you found in (a).

1. $a_n = 5 + 2(n - 1)$ **2.** $a_n = 3 - 4(n - 1)$
3. $a_n = \frac{5}{2} - (n - 1)$ **4.** $a_n = \frac{1}{2}(n - 1)$

5–8 ■ Find the nth term of the arithmetic sequence with given first term a and common difference d. What is the 10th term?

5. $a = 3, d = 5$ **6.** $a = -6, d = 3$
7. $a = \frac{5}{2}, d = -\frac{1}{2}$ **8.** $a = \sqrt{3}, d = \sqrt{3}$

9–16 ■ Determine whether the sequence is arithmetic. If it is arithmetic, find the common difference.

9. $5, 8, 11, 14, \ldots$ **10.** $3, 6, 9, 13, \ldots$
11. $2, 4, 8, 16, \ldots$ **12.** $2, 4, 6, 8, \ldots$
13. $3, \frac{3}{2}, 0, -\frac{3}{2}, \ldots$ **14.** $\ln 2, \ln 4, \ln 8, \ln 16, \ldots$
15. $2.6, 4.3, 6.0, 7.7, \ldots$ **16.** $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

17–22 ■ Find the first five terms of the sequence and determine if it is arithmetic. If it is arithmetic, find the common difference and express the nth term of the sequence in the standard form $a_n = a + (n - 1)d$.

17. $a_n = 4 + 7n$ **18.** $a_n = 4 + 2^n$
19. $a_n = \dfrac{1}{1 + 2n}$ **20.** $a_n = 1 + \dfrac{n}{2}$
21. $a_n = 6n - 10$ **22.** $a_n = 3 + (-1)^n n$

23–32 ■ Determine the common difference, the fifth term, the nth term, and the 100th term of the arithmetic sequence.

23. $2, 5, 8, 11, \ldots$ **24.** $1, 5, 9, 13, \ldots$
25. $4, 9, 14, 19, \ldots$ **26.** $11, 8, 5, 2, \ldots$
27. $-12, -8, -4, 0, \ldots$ **28.** $\frac{7}{6}, \frac{5}{3}, \frac{13}{6}, \frac{8}{3}, \ldots$
29. $25, 26.5, 28, 29.5, \ldots$ **30.** $15, 12.3, 9.6, 6.9, \ldots$
31. $2, 2 + s, 2 + 2s, 2 + 3s, \ldots$
32. $-t, -t + 3, -t + 6, -t + 9, \ldots$

33. The tenth term of an arithmetic sequence is $\frac{55}{2}$, and the second term is $\frac{7}{2}$. Find the first term.

34. The 12th term of an arithmetic sequence is 32, and the fifth term is 18. Find the 20th term.

35. The 100th term of an arithmetic sequence is 98, and the common difference is 2. Find the first three terms.

36. The 20th term of an arithmetic sequence is 101, and the common difference is 3. Find a formula for the nth term.

37. Which term of the arithmetic sequence $1, 4, 7, \ldots$ is 88?

38. The first term of an arithmetic sequence is 1, and the common difference is 4. Is 11,937 a term of this sequence? If so, which term is it?

39–44 ■ Find the partial sum S_n of the arithmetic sequence that satisfies the given conditions.

39. $a = 1, d = 2, n = 10$ **40.** $a = 3, d = 2, n = 12$
41. $a = 4, d = 2, n = 20$ **42.** $a = 100, d = -5, n = 8$
43. $a_1 = 55, d = 12, n = 10$ **44.** $a_2 = 8, a_5 = 9.5, n = 15$

45–50 ■ A partial sum of an arithmetic sequence is given. Find the sum.

45. $1 + 5 + 9 + \cdots + 401$
46. $-3 + \left(-\frac{3}{2}\right) + 0 + \frac{3}{2} + 3 + \cdots + 30$
47. $0.7 + 2.7 + 4.7 + \cdots + 56.7$
48. $-10 - 9.9 - 9.8 - \cdots - 0.1$
49. $\sum_{k=0}^{10} (3 + 0.25k)$ **50.** $\sum_{n=0}^{20} (1 - 2n)$

51. Show that a right triangle whose sides are in arithmetic progression is similar to a 3–4–5 triangle.

52. Find the product of the numbers

$$10^{1/10}, 10^{2/10}, 10^{3/10}, 10^{4/10}, \ldots, 10^{19/10}$$

53. A sequence is **harmonic** if the reciprocals of the terms of the sequence form an arithmetic sequence. Determine whether the following sequence is harmonic:

$$1, \tfrac{3}{5}, \tfrac{3}{7}, \tfrac{1}{3}, \ldots$$

54. The **harmonic mean** of two numbers is the reciprocal of the average of the reciprocals of the two numbers. Find the harmonic mean of 3 and 5.

55. An arithmetic sequence has first term $a = 5$ and common difference $d = 2$. How many terms of this sequence must be added to get 2700?

56. An arithmetic sequence has first term $a_1 = 1$ and fourth term $a_4 = 16$. How many terms of this sequence must be added to get 2356?

After the first term, the terms of this series form an infinite geometric series with

$$a = \frac{51}{1000} \quad \text{and} \quad r = \frac{1}{100}$$

Thus, the sum of this part of the series is

$$S = \frac{\frac{51}{1000}}{1 - \frac{1}{100}} = \frac{\frac{51}{1000}}{\frac{99}{100}} = \frac{51}{1000} \cdot \frac{100}{99} = \frac{51}{990}$$

So,

$$2.3\overline{51} = \frac{23}{10} + \frac{51}{990} = \frac{2328}{990} = \frac{388}{165}$$

11.3 Exercises

1–4 ■ The nth term of a sequence is given.
(a) Find the first five terms of the sequence.
(b) What is the common ratio r?
(c) Graph the terms you found in (a).

1. $a_n = 5(2)^{n-1}$ **2.** $a_n = 3(-4)^{n-1}$

3. $a_n = \frac{5}{2}(-\frac{1}{2})^{n-1}$ **4.** $a_n = 3^{n-1}$

5–8 ■ Find the nth term of the geometric sequence with given first term a and common ratio r. What is the fourth term?

5. $a = 3, \quad r = 5$ **6.** $a = -6, \quad r = 3$

7. $a = \frac{5}{2}, \quad r = -\frac{1}{2}$ **8.** $a = \sqrt{3}, \quad r = \sqrt{3}$

9–16 ■ Determine whether the sequence is geometric. If it is geometric, find the common ratio.

9. $2, 4, 8, 16, \ldots$ **10.** $2, 6, 18, 36, \ldots$

11. $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \ldots$ **12.** $27, -9, 3, -1, \ldots$

13. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ **14.** $e^2, e^4, e^6, e^8, \ldots$

15. $1.0, 1.1, 1.21, 1.331, \ldots$ **16.** $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$

17–22 ■ Find the first five terms of the sequence and determine if it is geometric. If it is geometric, find the common ratio and express the nth term of the sequence in the standard form $a_n = ar^{n-1}$.

17. $a_n = 2(3)^n$ **18.** $a_n = 4 + 3^n$

19. $a_n = \dfrac{1}{4^n}$ **20.** $a_n = (-1)^n 2^n$

21. $a_n = \ln(5^{n-1})$ **22.** $a_n = n^n$

23–32 ■ Determine the common ratio, the fifth term, and the nth term of the geometric sequence.

23. $2, 6, 18, 54, \ldots$ **24.** $7, \frac{14}{3}, \frac{28}{9}, \frac{56}{27}, \ldots$

25. $0.3, -0.09, 0.027, -0.0081, \ldots$

26. $1, \sqrt{2}, 2, 2\sqrt{2}, \ldots$

27. $144, -12, 1, -\frac{1}{12}, \ldots$ **28.** $-8, -2, -\frac{1}{2}, -\frac{1}{8}, \ldots$

29. $3, 3^{5/3}, 3^{7/3}, 27, \ldots$ **30.** $t, \dfrac{t^2}{2}, \dfrac{t^3}{4}, \dfrac{t^4}{8}, \ldots$

31. $1, s^{2/7}, s^{4/7}, s^{6/7}, \ldots$ **32.** $5, 5^{c+1}, 5^{2c+1}, 5^{3c+1}, \ldots$

33. The first term of a geometric sequence is 8, and the secon term is 4. Find the fifth term.

34. The first term of a geometric sequence is 3, and the third term is $\frac{4}{3}$. Find the fifth term.

35. The common ratio in a geometric sequence is $\frac{2}{5}$, and the fourth term is $\frac{5}{2}$. Find the third term.

36. The common ratio in a geometric sequence is $\frac{3}{2}$, and the f term is 1. Find the first three terms.

37. Which term of the geometric sequence $2, 6, 18, \ldots$ is 118,098?

38. The second and the fifth terms of a geometric sequence a: 10 and 1250, respectively. Is 31,250 a term of this sequen If so, which term is it?

39–42 ■ Find the partial sum S_n of the geometric sequence tl satisfies the given conditions.

39. $a = 5, \quad r = 2, \quad n = 6$ **40.** $a = \frac{2}{3}, \quad r = \frac{1}{3}, \quad n = 4$

41. $a_3 = 28, \quad a_6 = 224, \quad n = 6$

42. $a_2 = 0.12, \quad a_5 = 0.00096, \quad n = 4$

43–46 ■ Find the sum.

43. $1 + 3 + 9 + \cdots + 2187$

44. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots - \frac{1}{512}$

45. $\displaystyle\sum_{k=0}^{10} 3(\tfrac{1}{2})^k$ **46.** $\displaystyle\sum_{j=0}^{5} 7(\tfrac{3}{2})^j$

47–54 ■ Find the sum of the infinite geometric series.

47. $1 + \dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \cdots$ **48.** $1 - \dfrac{1}{2} + \dfrac{1}{4} - \dfrac{1}{8} + \cdots$

49. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$

50. $\frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \cdots$

51. $\frac{1}{3^6} + \frac{1}{3^8} + \frac{1}{3^{10}} + \frac{1}{3^{12}} + \cdots$

52. $3 - \frac{3}{2} + \frac{3}{4} - \frac{3}{8} + \cdots$

53. $-\frac{100}{9} + \frac{10}{3} - 1 + \frac{3}{10} - \cdots$

54. $\frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}} + \frac{1}{4} + \cdots$

55–60 ■ Express the repeating decimal as a fraction.

55. $0.777\ldots$

56. $0.2\overline{53}$

57. $0.030303\ldots$

58. $2.11\overline{25}$

59. $0.\overline{112}$

60. $0.123123123\ldots$

61. If the numbers $a_1, a_2, \ldots, a_n$ form a geometric sequence, then $a_2, a_3, \ldots, a_{n-1}$ are **geometric means** between a_1 and a_n. Insert three geometric means between 5 and 80.

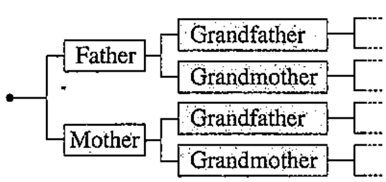
62. Find the sum of the first ten terms of the sequence

$$a + b, a^2 + 2b, a^3 + 3b, a^4 + 4b, \ldots$$

Applications

63. **Depreciation** A construction company purchases a bulldozer for \$160,000. Each year the value of the bulldozer depreciates by 20% of its value in the preceding year. Let V_n be the value of the bulldozer in the nth year. (Let $n = 1$ be the year the bulldozer is purchased.)
   (a) Find a formula for V_n.
   (b) In what year will the value of the bulldozer be less than \$100,000?

64. **Family Tree** A person has two parents, four grandparents, eight great-grandparents, and so on. How many ancestors does a person have 15 generations back?



65. **Bouncing Ball** A ball is dropped from a height of 80 ft. The elasticity of this ball is such that it rebounds three-fourths of the distance it has fallen. How high does the ball rebound on the fifth bounce? Find a formula for how high the ball rebounds on the nth bounce.

66. **Bacteria Culture** A culture initially has 5000 bacteria, and its size increases by 8% every hour. How many bacteria are present at the end of 5 hours? Find a formula for the number of bacteria present after n hours.

67. **Mixing Coolant** A truck radiator holds 5 gal and is filled with water. A gallon of water is removed from the radiator and replaced with a gallon of antifreeze; then, a gallon of the mixture is removed from the radiator and again replaced by a gallon of antifreeze. This process is repeated indefinitely. How much water remains in the tank after this process is repeated 3 times? 5 times? n times?

68. **Musical Frequencies** The frequencies of musical notes (measured in cycles per second) form a geometric sequence. Middle C has a frequency of 256, and the C that is an octave higher has a frequency of 512. Find the frequency of C two octaves below middle C.

69. **Bouncing Ball** A ball is dropped from a height of 9 ft. The elasticity of the ball is such that it always bounces up one-third the distance it has fallen.
   (a) Find the total distance the ball has traveled at the instant it hits the ground the fifth time.
   (b) Find a formula for the total distance the ball has traveled at the instant it hits the ground the nth time.

70. **Geometric Savings Plan** A very patient woman wishes to become a billionaire. She decides to follow a simple scheme: She puts aside 1 cent the first day, 2 cents the second day, 4 cents the third day, and so on, doubling the number of cents each day. How much money will she have at the end of 30 days? How many days will it take this woman to realize her wish?

71. **St. Ives** The following is a well-known children's rhyme:

As I was going to St. Ives
I met a man with seven wives;
Every wife had seven sacks;
Every sack had seven cats;
Every cat had seven kits;
Kits, cats, sacks, and wives,
How many were going to St. Ives?

Assuming that the entire group is actually going to St. Ives, show that the answer to the question in the rhyme is a partial sum of a geometric sequence, and find the sum.

72. **Drug Concentration** A certain drug is administered once a day. The concentration of the drug in the patient's bloodstream increases rapidly at first, but each successive dose has less effect than the preceding one. The total amount of the drug (in mg) in the bloodstream after the nth dose is given by

$$\sum_{k=1}^{n} 50\left(\tfrac{1}{2}\right)^{k-1}$$

Solution Let $P(n)$ denote the statement $4n < 2^n$.

Step 1 $P(5)$ is the statement that $4 \cdot 5 < 2^5$, or $20 < 32$, which is true.

Step 2 Assume that $P(k)$ is true. Thus, our induction hypothesis is

$$4k < 2^k$$

We want to use this to show that $P(k+1)$ is true, that is,

$$4(k+1) < 2^{k+1}$$

We get $P(k+1)$ by replacing k by $k+1$ in the statement $P(k)$.

So, we start with the left side of the inequality and use the induction hypothesis to show that it is less than the right side. For $k \geq 5$, we have

$$\begin{aligned} 4(k+1) &= 4k + 4 \\ &< 2^k + 4 && \text{Induction hypothesis} \\ &< 2^k + 4k && \text{Because } 4 < 4k \\ &< 2^k + 2^k && \text{Induction hypothesis} \\ &= 2 \cdot 2^k \\ &= 2^{k+1} && \text{Property of exponents} \end{aligned}$$

Thus, $P(k+1)$ follows from $P(k)$ and this completes the induction step.

Having proved Steps 1 and 2, we conclude by the Principle of Mathematical Induction that $P(n)$ is true for all natural numbers $n \geq 5$. ■

11.5 Exercises

1–12 ■ Use mathematical induction to prove that the formula is true for all natural numbers n.

1. $2 + 4 + 6 + \cdots + 2n = n(n+1)$
2. $1 + 4 + 7 + \cdots + (3n-2) = \dfrac{n(3n-1)}{2}$
3. $5 + 8 + 11 + \cdots + (3n+2) = \dfrac{n(3n+7)}{2}$
4. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$
5. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$
6. $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \dfrac{n(n+1)(2n+7)}{6}$
7. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$
8. $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2-1)$
9. $2^3 + 4^3 + 6^3 + \cdots + (2n)^3 = 2n^2(n+1)^2$
10. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{(n+1)}$
11. $1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \cdots + n \cdot 2^n = 2[1 + (n-1)2^n]$
12. $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$
13. Show that $n^2 + n$ is divisible by 2 for all natural numbers n.
14. Show that $5^n - 1$ is divisible by 4 for all natural numbers n.
15. Show that $n^2 - n + 41$ is odd for all natural numbers n.
16. Show that $n^3 - n + 3$ is divisible by 3 for all natural numbers n.
17. Show that $8^n - 3^n$ is divisible by 5 for all natural numbers n.
18. Show that $3^{2n} - 1$ is divisible by 8 for all natural numbers n.
19. Prove that $n < 2^n$ for all natural numbers n.
20. Prove that $(n+1)^2 < 2n^2$ for all natural numbers $n \geq 3$.
21. Prove that if $x > -1$, then $(1+x)^n \geq 1 + nx$ for all natural numbers n.
22. Show that $100n \leq n^2$ for all $n \geq 100$.
23. Let $a_{n+1} = 3a_n$ and $a_1 = 5$. Show that $a_n = 5 \cdot 3^{n-1}$ for all natural numbers n.

We use this to show that $P(k+1)$ is true.

$$(a+b)^{k+1} = (a+b)[(a+b)^k]$$

$$= (a+b)\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k\right]$$ Induction hypothesis

$$= a\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k\right]$$

$$+ b\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k\right]$$ Distributive Property

$$= \binom{k}{0}a^{k+1} + \binom{k}{1}a^kb + \binom{k}{2}a^{k-1}b^2 + \cdots + \binom{k}{k-1}a^2b^{k-1} + \binom{k}{k}ab^k$$

$$+ \binom{k}{0}a^kb + \binom{k}{1}a^{k-1}b^2 + \binom{k}{2}a^{k-2}b^3 + \cdots + \binom{k}{k-1}ab^k + \binom{k}{k}b^{k+1}$$ Distributive Property

$$= \binom{k}{0}a^{k+1} + \left[\binom{k}{0} + \binom{k}{1}\right]a^kb + \left[\binom{k}{1} + \binom{k}{2}\right]a^{k-1}b^2$$

$$+ \cdots + \left[\binom{k}{k-1} + \binom{k}{k}\right]ab^k + \binom{k}{k}b^{k+1}$$ Group like

Using the key property of the binomial coefficients, we can write each (expressions in square brackets as a single binomial coefficient. Also, w the first and last coefficients as $\binom{k+1}{0}$ and $\binom{k+1}{k+1}$ (these are equal to 1 by cise 46) gives

$$(a+b)^{k+1} = \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^kb + \binom{k+1}{2}a^{k-1}b^2 + \cdots + \binom{k+1}{k}ab^k + \binom{k+1}{k+1}$$

But this last equation is precisely $P(k+1)$, and this completes the i tion step.

Having proved Steps 1 and 2, we conclude by the Principle of Mathem Induction that the theorem is true for all natural numbers n.

11.6 Exercises

1–12 ■ Use Pascal's triangle to expand the expression.

1. $(x+y)^6$ **2.** $(2x+1)^4$ **3.** $\left(x+\frac{1}{x}\right)^4$

4. $(x-y)^5$ **5.** $(x-1)^5$ **6.** $(\sqrt{a}+\sqrt{b})^6$

7. $(x^2y-1)^5$ **8.** $(1+\sqrt{2})^6$ **9.** $(2x-3y)^3$

10. $(1+x^3)^3$ **11.** $\left(\frac{1}{x}-\sqrt{x}\right)^5$ **12.** $\left(2+\frac{x}{2}\right)^5$

13–20 ■ Evaluate the expression.

13. $\binom{6}{4}$ **14.** $\binom{8}{3}$ **15.** $\binom{100}{98}$

16. $\binom{10}{5}$ **17.** $\binom{3}{1}\binom{4}{2}$ **18.** $\binom{5}{2}($

19. $\binom{5}{0}+\binom{5}{1}+\binom{5}{2}+\binom{5}{3}+\binom{5}{4}+\binom{5}{5}$

20. $\binom{5}{0} - \binom{5}{1} + \binom{5}{2} - \binom{5}{3} + \binom{5}{4} - \binom{5}{5}$

21–24 ■ Use the Binomial Theorem to expand the expression.

21. $(x + 2y)^4$

22. $(1 - x)^5$

23. $\left(1 + \dfrac{1}{x}\right)^6$

24. $(2A + B^2)^4$

25. Find the first three terms in the expansion of $(x + 2y)^{20}$.

26. Find the first four terms in the expansion of $(x^{1/2} + 1)^{30}$.

27. Find the last two terms in the expansion of $(a^{2/3} + a^{1/3})^{25}$.

28. Find the first three terms in the expansion of

$$\left(x + \frac{1}{x}\right)^{40}$$

29. Find the middle term in the expansion of $(x^2 + 1)^{18}$.

30. Find the fifth term in the expansion of $(ab - 1)^{20}$.

31. Find the 24th term in the expansion of $(a + b)^{25}$.

32. Find the 28th term in the expansion of $(A - B)^{30}$.

33. Find the 100th term in the expansion of $(1 + y)^{100}$.

34. Find the second term in the expansion of

$$\left(x^2 - \frac{1}{x}\right)^{25}$$

35. Find the term containing x^4 in the expansion of $(x + 2y)^{10}$.

36. Find the term containing y^3 in the expansion of $(\sqrt{2} + y)^{12}$.

37. Find the term containing b^8 in the expansion of $(a + b^2)^{12}$.

38. Find the term that does not contain x in the expansion of

$$\left(8x + \frac{1}{2x}\right)^8$$

39–42 ■ Factor using the Binomial Theorem.

39. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

40. $(x - 1)^5 + 5(x - 1)^4 + 10(x - 1)^3 + 10(x - 1)^2 + 5(x - 1) + 1$

41. $8a^3 + 12a^2b + 6ab^2 + b^3$

42. $x^8 + 4x^6y + 6x^4y^2 + 4x^2y^3 + y^4$

43–44 ■ Simplify using the Binomial Theorem.

43. $\dfrac{(x + h)^3 - x^3}{h}$

44. $\dfrac{(x + h)^4 - x^4}{h}$

45. Show that $(1.01)^{100} > 2$.
[*Hint:* Note that $(1.01)^{100} = (1 + 0.01)^{100}$ and use the Binomial Theorem to show that the sum of the first two terms of the expansion is greater than 2.]

46. Show that $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$.

47. Show that $\binom{n}{1} = \binom{n}{n-1} = n$.

48. Show that $\binom{n}{r} = \binom{n}{n-r}$ for $0 \le r \le n$.

49. In this exercise we prove the identity

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

(a) Write the left side of this equation as the sum of two fractions.
(b) Show that a common denominator of the expression you found in part (a) is $r!(n - r + 1)!$.
(c) Add the two fractions using the common denominator in part (b), simplify the numerator, and note that the resulting expression is equal to the right side of the equation.

50. Prove that $\binom{n}{r}$ is an integer for all n and for $0 \le r \le n$.
[*Suggestion:* Use induction to show that the statement is true for all n, and use Exercise 49 for the induction step.]

Discovery • Discussion

51. **Powers of Factorials** Which is larger, $(100!)^{101}$ or $(101!)^{100}$? [*Hint:* Try factoring the expressions. Do they have any common factors?]

52. **Sums of Binomial Coefficients** Add each of the first five rows of Pascal's triangle, as indicated. Do you see a pattern?

$$1 + 1 = \boxed{?}$$
$$1 + 2 + 1 = \boxed{?}$$
$$1 + 3 + 3 + 1 = \boxed{?}$$
$$1 + 4 + 6 + 4 + 1 = \boxed{?}$$
$$1 + 5 + 10 + 10 + 5 + 1 = \boxed{?}$$

s n.

2

· all

$B)^{10}$.

11 Test

1. Find the first four terms and the tenth term of the sequence whose nth term is $a_n = n^2 - 1$.
2. A sequence is defined recursively by $a_{n+2} = a_n^2 - a_{n+1}$, with $a_1 = 1$ and $a_2 = 1$. Find a_5.
3. An arithmetic sequence begins 2, 5, 8, 11, 14, ...
   (a) Find the common difference d for this sequence.
   (b) Find a formula for the nth term a_n of the sequence.
   (c) Find the 35th term of the sequence.
4. A geometric sequence begins 12, 3, 3/4, 3/16, 3/64, ...
   (a) Find the common ratio r for this sequence.
   (b) Find a formula for the nth term a_n of the sequence.
   (c) Find the tenth term of the sequence.
5. The first term of a geometric sequence is 25, and the fourth term is $\frac{1}{5}$.
   (a) Find the common ratio r and the fifth term.
   (b) Find the partial sum of the first eight terms.
6. The first term of an arithmetic sequence is 10 and the tenth term is 2.
   (a) Find the common difference and the 100th term of the sequence.
   (b) Find the partial sum of the first ten terms.
7. Let $a_1, a_2, a_3, \ldots$ be a geometric sequence with initial term a and common ratio r. Show that $a_1^2, a_2^2, a_3^2, \ldots$ is also a geometric sequence by finding its common ratio.
8. Write the expression without using sigma notation, and then find the sum.
   (a) $\sum_{n=1}^{5} (1 - n^2)$ (b) $\sum_{n=3}^{6} (-1)^n 2^{n-2}$
9. Find the sum.
   (a) $\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \frac{2^3}{3^4} + \cdots + \frac{2^9}{3^{10}}$
   (b) $1 + \frac{1}{2^{1/2}} + \frac{1}{2} + \frac{1}{2^{3/2}} + \cdots$
10. Use mathematical induction to prove that, for all natural numbers n,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

11. Expand $(2x + y^2)^5$.
12. Find the term containing x^3 in the binomial expansion of $(3x - 2)^{10}$.
13. A puppy weighs 0.85 lb at birth, and each week he gains 24% in weight. Let a_n be his weight in pounds at the end of his nth week of life.
    (a) Find a formula for a_n.
    (b) How much does the puppy weigh when he is six weeks old?
    (c) Is the sequence $a_1, a_2, a_3, \ldots$ arithmetic, geometric, or neither?

12.1 Exercises

1–6 ■ Complete the table of values (to five decimal places) and use the table to estimate the value of the limit.

1. $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$						

2. $\lim_{x\to 2} \frac{x-2}{x^2+x-6}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

3. $\lim_{x\to 1} \frac{x-1}{x^3-1}$

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$						

4. $\lim_{x\to 0} \frac{e^x-1}{x}$

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$						

5. $\lim_{x\to 0} \frac{\sin x}{x}$

x	±1	±0.5	±0.1	±0.05	±0.01
$f(x)$					

6. $\lim_{x\to 0^+} x \ln x$

x	0.1	0.01	0.001	0.0001	0.00001
$f(x)$					

7–12 ■ Use a table of values to estimate the value of the limit. Then use a graphing device to confirm your result graphically.

7. $\lim_{x\to -4} \frac{x+4}{x^2+7x+12}$

8. $\lim_{x\to 1} \frac{x^3-1}{x^2-1}$

9. $\lim_{x\to 0} \frac{5^x-3^x}{x}$

10. $\lim_{x\to 0} \frac{\sqrt{x+9}-3}{x}$

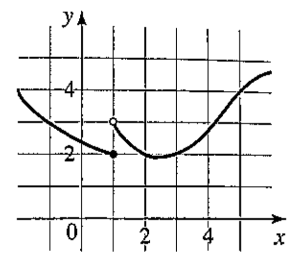
11. $\lim_{x\to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

12. $\lim_{x\to 0} \frac{\tan 2x}{\tan 3x}$

13. For the function f whose graph is given, state the value of the given quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x\to 1^-} f(x)$ (b) $\lim_{x\to 1^+} f(x)$ (c) $\lim_{x\to 1} f(x)$

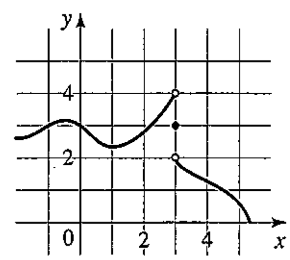
(d) $\lim_{x\to 5} f(x)$ (e) $f(5)$



14. For the function f whose graph is given, state the value of the given quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x\to 0} f(x)$ (b) $\lim_{x\to 3^-} f(x)$ (c) $\lim_{x\to 3^+} f(x)$

(d) $\lim_{x\to 3} f(x)$ (e) $f(3)$

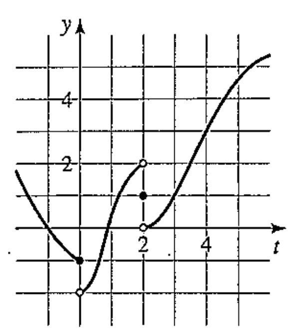


15. For the function g whose graph is given, state the value of the given quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{t\to 0^-} g(t)$ (b) $\lim_{t\to 0^+} g(t)$ (c) $\lim_{t\to 0} g(t)$

(d) $\lim_{t\to 2^-} g(t)$ (e) $\lim_{t\to 2^+} g(t)$ (f) $\lim_{t\to 2} g(t)$

(g) $g(2)$ (h) $\lim_{t\to 4} g(t)$



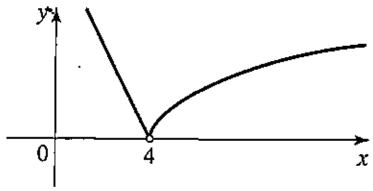


Figure 4

Since $f(x) = 8 - 2x$ for $x < 4$, we have

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (8 - 2x) = 8 - 2 \cdot 4 = 0$$

The right- and left-hand limits are equal. Thus, the limit exists and

$$\lim_{x \to 4} f(x) = 0$$

The graph of f is shown in Figure 4. ■

12.2 Exercises

1. Suppose that

$$\lim_{x \to a} f(x) = -3 \qquad \lim_{x \to a} g(x) = 0 \qquad \lim_{x \to a} h(x) = 8$$

Find the value of the given limit. If the limit does not exist, explain why.

(a) $\lim_{x \to a} [f(x) + h(x)]$ (b) $\lim_{x \to a} [f(x)]^2$

(c) $\lim_{x \to a} \sqrt[3]{h(x)}$ (d) $\lim_{x \to a} \dfrac{1}{f(x)}$

(e) $\lim_{x \to a} \dfrac{f(x)}{h(x)}$ (f) $\lim_{x \to a} \dfrac{g(x)}{f(x)}$

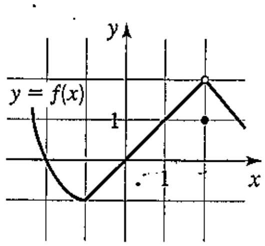
(g) $\lim_{x \to a} \dfrac{f(x)}{g(x)}$ (h) $\lim_{x \to a} \dfrac{2f(x)}{h(x) - f(x)}$

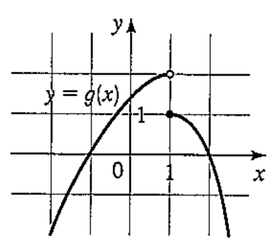
2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.

(a) $\lim_{x \to 2} [f(x) + g(x)]$ (b) $\lim_{x \to 1} [f(x) + g(x)]$

(c) $\lim_{x \to 0} [f(x)g(x)]$ (d) $\lim_{x \to -1} \dfrac{f(x)}{g(x)}$

(e) $\lim_{x \to 2} x^3 f(x)$ (f) $\lim_{x \to 1} \sqrt{3 + f(x)}$





3–8 ■ Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\lim_{x \to 4} (5x^2 - 2x + 3)$ **4.** $\lim_{x \to 3} (x^3 + 2)(x^2 - 5x)$

5. $\lim_{x \to -1} \dfrac{x - 2}{x^2 + 4x - 3}$ **6.** $\lim_{x \to 1} \left(\dfrac{x^4 + x^2 - 6}{x^4 + 2x + 3} \right)^2$

7. $\lim_{t \to -2} (t + 1)^9 (t^2 - 1)$ **8.** $\lim_{u \to -2} \sqrt{u^4 + 3u + 6}$

9–20 ■ Evaluate the limit, if it exists.

9. $\lim_{x \to 2} \dfrac{x^2 + x - 6}{x - 2}$ **10.** $\lim_{x \to -4} \dfrac{x^2 + 5x + 4}{x^2 + 3x - 4}$

11. $\lim_{x \to 2} \dfrac{x^2 - x + 6}{x + 2}$ **12.** $\lim_{x \to 1} \dfrac{x^3 - 1}{x^2 - 1}$

13. $\lim_{t \to -3} \dfrac{t^2 - 9}{2t^2 + 7t + 3}$ **14.** $\lim_{h \to 0} \dfrac{\sqrt{1 + h} - 1}{h}$

15. $\lim_{h \to 0} \dfrac{(2 + h)^3 - 8}{h}$ **16.** $\lim_{x \to 2} \dfrac{x^4 - 16}{x - 2}$

17. $\lim_{x \to 7} \dfrac{\sqrt{x + 2} - 3}{x - 7}$ **18.** $\lim_{h \to 0} \dfrac{(3 + h)^{-1} - 3^{-1}}{h}$

19. $\lim_{x \to -4} \dfrac{\frac{1}{4} + \frac{1}{x}}{4 + x}$ **20.** $\lim_{t \to 0} \left(\dfrac{1}{t} - \dfrac{1}{t^2 + t} \right)$

21–24 ■ Find the limit and use a graphing device to confirm your result graphically.

21. $\lim_{x \to 1} \dfrac{x^2 - 1}{\sqrt{x} - 1}$ **22.** $\lim_{x \to 0} \dfrac{(4 + x)^3 - 64}{x}$

23. $\lim_{x \to -1} \dfrac{x^2 - x - 2}{x^3 - x}$ **24.** $\lim_{x \to 1} \dfrac{x^8 - 1}{x^5 - x}$

25. (a) Estimate the value of

$$\lim_{x \to 0} \frac{x}{\sqrt{1 + 3x} - 1}$$

by graphing the function $f(x) = x/(\sqrt{1 + 3x} - 1)$.

(b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.

26. (a) Use a graph of

$$f(x) = \frac{\sqrt{3 + x} - \sqrt{3}}{x}$$

to estimate the value of $\lim_{x \to 0} f(x)$ to two decimal places.

Now we can compute the limit:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5}{2}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \qquad \text{Definition of } a_n$$

$$= \frac{5}{2} \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)\lim_{n\to\infty}\left(2 + \frac{1}{n}\right) \qquad \text{Limit of a Product}$$

$$= \frac{5}{2}(1)(2) = 5 \qquad \text{Let } n \to \infty$$

■

12.4 Exercises

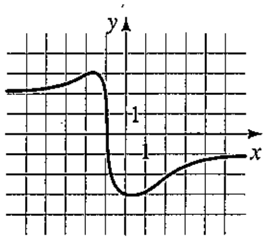
1–2 ■ (a) Use the graph of f to find the following limits.

(i) $\lim_{x\to\infty} f(x)$

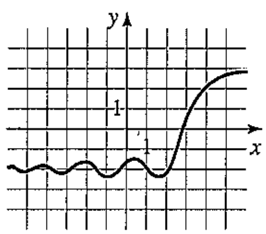
(ii) $\lim_{x\to -\infty} f(x)$

(b) State the equations of the horizontal asymptotes.

1.



2.



3–14 ■ Find the limit.

3. $\lim_{x\to\infty} \frac{6}{x}$

4. $\lim_{x\to\infty} \frac{3}{x^4}$

5. $\lim_{x\to\infty} \frac{2x + 1}{5x - 1}$

6. $\lim_{x\to\infty} \frac{2 - 3x}{4x + 5}$

7. $\lim_{x\to -\infty} \frac{4x^2 + 1}{2 + 3x^2}$

8. $\lim_{x\to -\infty} \frac{x^2 + 2}{x^3 + x + 1}$

9. $\lim_{t\to\infty} \frac{8t^3 + t}{(2t - 1)(2t^2 + 1)}$

10. $\lim_{r\to\infty} \frac{4r^3 - r^2}{(r + 1)^3}$

11. $\lim_{x\to\infty} \frac{x^4}{1 - x^2 + x^3}$

12. $\lim_{t\to\infty} \left(\frac{1}{t} - \frac{2t}{t - 1}\right)$

13. $\lim_{x\to -\infty} \left(\frac{x - 1}{x + 1} + 6\right)$

14. $\lim_{x\to\infty} \cos x$

15–18 ■ Use a table of values to estimate the limit. Then use a graphing device to confirm your result graphically.

15. $\lim_{x\to -\infty} \frac{\sqrt{x^2 + 4x}}{4x + 1}$

16. $\lim_{x\to\infty} \left(\sqrt{9x^2 + x} - 3x\right)$

17. $\lim_{x\to\infty} \frac{x^5}{e^x}$

18. $\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^{3x}$

19–30 ■ If the sequence is convergent, find its limit. If it is divergent, explain why.

19. $a_n = \frac{1 + n}{n + n^2}$

20. $a_n = \frac{5n}{n + 5}$

21. $a_n = \frac{n^2}{n + 1}$

22. $a_n = \frac{n - 1}{n^3 + 1}$

23. $a_n = \frac{1}{3^n}$

24. $a_n = \frac{(-1)^n}{n}$

25. $a_n = \sin(n\pi/2)$

26. $a_n = \cos n\pi$

27. $a_n = \frac{3}{n^2}\left[\frac{n(n + 1)}{2}\right]$

28. $a_n = \frac{5}{n}\left(n + \frac{4}{n}\left[\frac{n(n + 1)}{2}\right]\right)$

29. $a_n = \frac{24}{n^3}\left[\frac{n(n + 1)(2n + 1)}{6}\right]$

30. $a_n = \frac{12}{n^4}\left[\frac{n(n + 1)}{2}\right]^2$

Applications

31. Salt Concentration

(a) A tank contains 5000 L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min. Show that the concentration of salt after t minutes (in grams per liter) is

$$C(t) = \frac{30t}{200 + t}$$

(b) What happens to the concentration as $t \to \infty$?

12.3 Exercises

1–6 ■ Find the slope of the tangent line to the graph of f at the given point.

1. $f(x) = 3x + 4$ at $(1, 7)$
2. $f(x) = 5 - 2x$ at $(-3, 11)$
3. $f(x) = 4x^2 - 3x$ at $(-1, 7)$
4. $f(x) = 1 + 2x - 3x^2$ at $(1, 0)$
5. $f(x) = 2x^3$ at $(2, 16)$
6. $f(x) = \dfrac{6}{x + 1}$ at $(2, 2)$

7–12 ■ Find an equation of the tangent line to the curve at the given point. Graph the curve and the tangent line.

7. $y = x + x^2$ at $(-1, 0)$
8. $y = 2x - x^3$ at $(1, 1)$
9. $y = \dfrac{x}{x - 1}$ at $(2, 2)$
10. $y = \dfrac{1}{x^2}$ at $(-1, 1)$
11. $y = \sqrt{x + 3}$ at $(1, 2)$
12. $y = \sqrt{1 + 2x}$ at $(4, 3)$

13–18 ■ Find the derivative of the function at the given number.

13. $f(x) = 1 - 3x^2$ at 2
14. $f(x) = 2 - 3x + x^2$ at -1
15. $g(x) = x^4$ at 1
16. $g(x) = 2x^2 + x^3$ at 1
17. $F(x) = \dfrac{1}{\sqrt{x}}$ at 4
18. $G(x) = 1 + 2\sqrt{x}$ at 4

19–22 ■ Find $f'(a)$, where a is in the domain of f.

19. $f(x) = x^2 + 2x$
20. $f(x) = -\dfrac{1}{x^2}$
21. $f(x) = \dfrac{x}{x + 1}$
22. $f(x) = \sqrt{x - 2}$

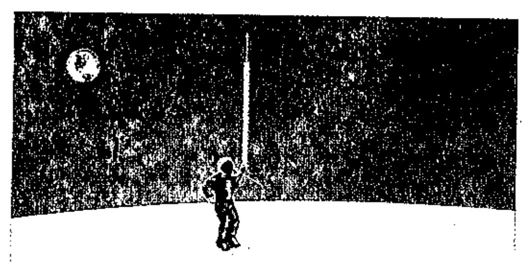
23. (a) If $f(x) = x^3 - 2x + 4$, find $f'(a)$.
(b) Find equations of the tangent lines to the graph of f at the points whose x-coordinates are 0, 1, and 2.
(c) Graph f and the three tangent lines.

24. (a) If $g(x) = 1/(2x - 1)$, find $g'(a)$.
(b) Find equations of the tangent lines to the graph of g at the points whose x-coordinates are -1, 0, and 1.
(c) Graph g and the three tangent lines.

Applications

25. **Velocity of a Ball** If a ball is thrown into the air with a velocity of 40 ft/s, its height (in feet) after t seconds is given by $y = 40t - 16t^2$. Find the velocity when $t = 2$.

26. **Velocity on the Moon** If an arrow is shot upward on moon with a velocity of 58 m/s, its height (in meters) af seconds is given by $H = 58t - 0.83t^2$.
(a) Find the velocity of the arrow after one second.
(b) Find the velocity of the arrow when $t = a$.
(c) At what time t will the arrow hit the moon?
(d) With what velocity will the arrow hit the moon?



27. **Velocity of a Particle** The displacement s (in meters) a particle moving in a straight line is given by the equatic of motion $s = 4t^3 + 6t + 2$, where t is measured in secor Find the velocity of the particle s at times $t = a$, $t = 1$, $t = 2$, $t = 3$.

28. **Inflating a Balloon** A spherical balloon is being inflat Find the rate of change of the surface area ($S = 4\pi r^2$) wi respect to the radius r when $r = 2$ ft.

29. **Temperature Change** A roast turkey is taken from an oven when its temperature has reached 185°F and is plac on a table in a room where the temperature is 75°F. The graph shows how the temperature of the turkey decreases

3. Describe several ways in which a limit can fail to exist. Illustrate with sketches.

4. State the following Limit Laws.
(a) Sum Law
(b) Difference Law
(c) Constant Multiple Law
(d) Product Law
(e) Quotient Law
(f) Power Law
(g) Root Law

5. Write an expression for the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.

6. Define the derivative $f'(a)$. Discuss two ways of interpreting this number.

7. If $y = f(x)$, write expressions for the following.
(a) The average rate of change of y with respect to x between the numbers a and x.
(b) The instantaneous rate of change of y with respect to x at $x = a$.

8. Explain the meaning of the equation

$$\lim_{x\to\infty} f(x) = 2$$

Draw sketches to illustrate the various possibilities.

9. (a) What does it mean to say that the line $y = L$ is a h zontal asymptote of the curve $y = f(x)$? Draw cu to illustrate the various possibilities.
(b) Which of the following curves have horizontal asymptotes?
(i) $y = x^2$ (iv) $y = \tan^{-1} x$
(ii) $y = 1/x$ (v) $y = e^x$
(iii) $y = \sin x$ (vi) $y = \ln x$

10. (a) What is a convergent sequence?
(b) What does $\lim_{n\to\infty} a_n = 3$ mean?

11. Suppose S is the region that lies under the graph of $y = f(x)$, $a \le x \le b$.
(a) Explain how this area is approximated using recta
(b) Write an expression for the area of S as a limit of sums.

Exercises

1–6 ■ Use a table of values to estimate the value of the limit. Then use a graphing device to confirm your result graphically.

1. $\displaystyle\lim_{x\to 2} \frac{x - 2}{x^2 - 3x + 2}$

2. $\displaystyle\lim_{t\to -1} \frac{t + 1}{t^3 - t}$

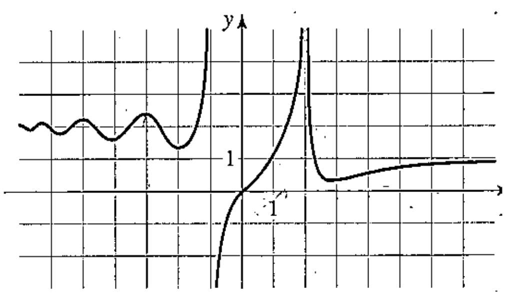
3. $\displaystyle\lim_{x\to 0} \frac{2^x - 1}{x}$

4. $\displaystyle\lim_{x\to 0} \frac{\sin 2x}{x}$

5. $\displaystyle\lim_{x\to 1^+} \ln\sqrt{x - 1}$

6. $\displaystyle\lim_{x\to 0^-} \frac{\tan x}{|x|}$

7. The graph of f is shown in the figure. Find each limit or explain why it does not exist.
(a) $\lim_{x\to 2^+} f(x)$ (b) $\lim_{x\to -3^+} f(x)$
(c) $\lim_{x\to -3^-} f(x)$ (d) $\lim_{x\to -3} f(x)$
(e) $\lim_{x\to 4} f(x)$ (f) $\lim_{x\to\infty} f(x)$
(g) $\lim_{x\to -\infty} f(x)$ (h) $\lim_{x\to 0} f(x)$



8. Let

$$f(x) = \begin{cases} 2 & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x \le 2 \\ x + 2 & \text{if } x > 2 \end{cases}$$

Find each limit or explain why it does not exist.
(a) $\lim_{x\to -1^-} f(x)$ (b) $\lim_{x\to -1^+} f(x)$
(c) $\lim_{x\to -1} f(x)$ (d) $\lim_{x\to 2^-} f(x)$

(e) $\lim_{x\to 2^+} f(x)$ (f) $\lim_{x\to 2} f(x)$

(g) $\lim_{x\to 0} f(x)$ (h) $\lim_{x\to 3} (f(x))^2$

9–20 ■ Use the Limit Laws to evaluate the limit, if it exists.

9. $\lim_{x\to 2} \dfrac{x+1}{x-3}$

10. $\lim_{t\to 1} (t^3 - 3t + 6)$

11. $\lim_{x\to 3} \dfrac{x^2 + x - 12}{x - 3}$

12. $\lim_{x\to -2} \dfrac{x^2 - 4}{x^2 + x - 2}$

13. $\lim_{u\to 0} \dfrac{(u+1)^2 - 1}{u}$

14. $\lim_{z\to 9} \dfrac{\sqrt{z} - 3}{z - 9}$

15. $\lim_{x\to 3^-} \dfrac{x - 3}{|x - 3|}$

16. $\lim_{x\to 0} \left(\dfrac{1}{x} + \dfrac{2}{x^2 - 2x}\right)$

17. $\lim_{x\to\infty} \dfrac{2x}{x - 4}$

18. $\lim_{x\to\infty} \dfrac{x^2 + 1}{x^4 - 3x + 6}$

19. $\lim_{x\to\infty} \cos^2 x$

20. $\lim_{t\to -\infty} \dfrac{t^4}{t^3 - 1}$

21–24 ■ Find the derivative of the function at the given number.

21. $f(x) = 3x - 5$, at 4

22. $g(x) = 2x^2 - 1$, at -1

23. $f(x) = \sqrt{x}$, at 16

24. $f(x) = \dfrac{x}{x+1}$, at 1

25–28 ■ (a) Find $f'(a)$. (b) Find $f'(2)$ and $f'(-2)$.

25. $f(x) = 6 - 2x$

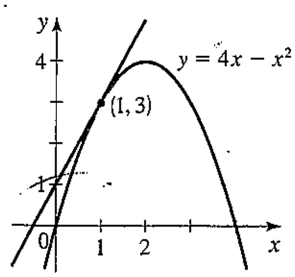
26. $f(x) = x^2 - 3x$

27. $f(x) = \sqrt{x + 6}$

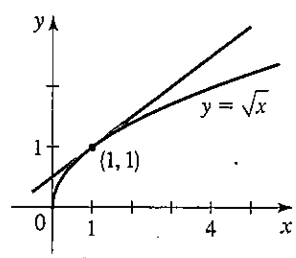
28. $f(x) = \dfrac{4}{x}$

29–30 ■ Find an equation of the tangent line shown in the figure.

29.



30.



31–34 ■ Find an equation of the line tangent to the graph of f at the given point.

31. $f(x) = 2x$, at $(3, 6)$

32. $f(x) = x^2 - 3$, at $(2, 1)$

33. $f(x) = \dfrac{1}{x}$, at $\left(2, \dfrac{1}{2}\right)$

34. $f(x) = \sqrt{x + 1}$, at $(3, 2)$

35. A stone is dropped from the roof of a building 640 ft above the ground. Its height (in feet) after t seconds is given by $h(t) = 640 - 16t^2$.

(a) Find the velocity of the stone when $t = 2$.

(b) Find the velocity of the stone when $t = a$.

(c) At what time t will the stone hit the ground?

(d) With what velocity will the stone hit the ground?

36. If a gas is confined in a fixed volume, then according to Boyle's Law the product of the pressure P and the temperature T is a constant. For a certain gas, $PT = 100$, where P is measured in lb/in^2 and T is measured in kelvins (K).

(a) Express P as a function of T.

(b) Find the instantaneous rate of change of P with respect to T when $T = 300$ K.

37–42 ■ If the sequence is convergent, find its limit. If it is divergent, explain why.

37. $a_n = \dfrac{n}{5n + 1}$

38. $a_n = \dfrac{n^3}{n^3 + 1}$

39. $a_n = \dfrac{n(n+1)}{2n^2}$

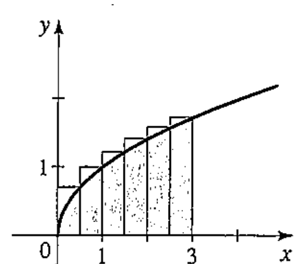
40. $a_n = \dfrac{n^3}{2n + 6}$

41. $a_n = \cos\left(\dfrac{n\pi}{2}\right)$

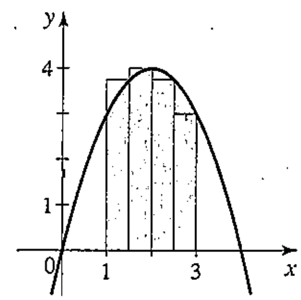
42. $a_n = \dfrac{10}{3^n}$

43–44 ■ Approximate the area of the shaded region under the graph of the given function by using the indicated rectangles. (The rectangles have equal width.)

43. $f(x) = \sqrt{x}$



44. $f(x) = 4x - x^2$



45–48 ■ Use the limit definition of area to find the area of the region that lies under the graph of f over the given interval.

45. $f(x) = 2x + 3$, $0 \le x \le 2$

46. $f(x) = x^2 + 1$, $0 \le x \le 3$

47. $f(x) = x^2 - x$, $1 \le x \le 2$

48. $f(x) = x^3$, $1 \le x \le 2$